ADVANCES IN MATHEMATICS RESEARCH

VOLUME 15

Albert R. Baswell
Editor

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CONTENTS

PREFACE

”Advances in Mathematics Research” presents original studies on the leading edge of mathematics. Each article has been carefully selected in an attempt to present substantial research results across a broad spectrum. Topics discussed herein include recent advances in the periodicity in dynamical systems; nonlinear differential equations and Fucik Spectrum; matrix theory; column and row determinants in quaternion linear algebra; elliptic perturbations of parabolic and hyperbolic problems and a discrete time (s,S) inventory system with service facility. (Imprint: Nova)

Periodic solutions and related notions of recurrence, invariance, limit sets and associated decompositions have been and remain among the most important topics in the theory and applications of differential equations and dynamical systems. Chapter 1 comprises a survey of some fundamental and recent advances along with several important open problems in these areas.

The topics discussed are the following: First, uniqueness of limit cycles for planar dynamical systems (differential equations) of Linard type, which is subsumed by Hilbert’s 16th problem on the number of limit cycles for planar systems (a fundamental problem of long standing that is still largely unresolved). The discussion revolves around a very general recent result of Zhou, Wang and Blackmore that subsumes virtually all extant theorems on uniqueness for Linard systems. Next, the focus is on the use of variational, geometric and topological methods for estimating the number of periodic solutions of Hamiltonian systems. Several recent results of Blackmore and Wang are described within the context of the considerable body of known results, and some related problems and research-in-progress are identified. Then, some advances in fixed point counts and persistence of invariant tori in Hamiltonian systems are surveyed via recent generalizations of Poincar-Birkhoff fixed point and KAM theorems, and several rather new and interesting applications of these results to problems in vortex dynamics are described. Finally, a brief characterization of ?-limit sets and its connections with recurrence is presented, where the approach emphasizes Conley theory and Morse decompositions. A new result is described and its relations to existing theorems, possible future research and open problems are treated in some detail.

Chapter 2 presents the development and evaluation of an approach to predict radon gas concentrations for unmeasured zip codes, using the Geographic Information System

(GIS) based interpolation techniques: kriging and cokriging. The radon gas concentration data collected by the county health departments, commercial testing services, university researchers, and the public between 1989 and 2008, for the state of Ohio, have been used to predict radon gas concentrations during the study. Note that monitoring radon gas concentration in houses across an entire state is very time consuming and involves huge investment.

Statistical performance measures, such as mean bias (MB), normalized mean square error (NMSE), coefficient of correlation (Corr.), factor of two (Fa2), fractional standard deviation (FS), and fractional bias (FB) have been used to assess the performance of interpolation schemes. Confidence limits for the measures of association (NMSE, Corr., and FB) have been obtained using the ”Bootstrap” method. The radon concentrations are over predicted (negative bias) by both of the interpolation techniques. On comparing the performance measures and the associated confidence limits on performance measures, it was observed that the cokriging interpolation technique had a slight edge over the kriging interpolation technique.

The zip code based results for radon gas concentrations exceeding 4 pCi/l have been tabulated using the cokriging interpolation technique for radon planners in Ohio. These results indicate that more work is needed to reduce radon gas concentrations in Ohio. The developed approach could be applied to any affected area of the globe.

Chapter 3 reports the results of a numerical experiment on the traveling salesman problem. The results indicate that, a large majority of instances of the problem is solvable within a practical time limit.

The goal of Chapter 4 is to put together some recent results concerning applications of monotone second order differential equations to singularly perturbed problems of elliptic - parabolic and elliptic - hyperbolic type. More exactly, the solution v of the heat equation or of the telegraph system is compared with the solution v_ε of an elliptic regularization. This elliptic regularization is a perturbed problem written with the aid of a small parameter $\varepsilon > 0$. It is a particular case of some second order differential equations governed by a maximal monotone operator in the Hilbert space $L^2(\Omega)$. Under some specific hypotheses, we construct a zero order asymptotic approximation for v_ε making use of the boundary layer function method of Vishik and Lyusternik. The higher order regularity of the solutions to both perturbed and unperturbed problems is investigated. The order of accuracy of the difference $v_\varepsilon - v$ is also established in some appropriate function spaces. Thus, the solution v of the heat equation (or telegraph system) is approximated by the solution v_ε of its elliptic regularization, which is a more regular function. This is a motivation for the study of the above mentioned second order evolution equations associated with monotone operators. This study can involve different unperturbed problems: semilinear heat equation, linear heat equation with nonlinear boundary conditions, semilinear telegraph system, nonlinear telegraph system with nonlinear boundary conditions, etc.

As discussed in Chapter 5, the diversity of problems involved in investigation of the interaction of hydrogen and its isotopes with solids is extensively covered in specialized literature [1]-[6]. In the context of hydrogen energy problems the interest in hydrides arises mainly from the following. First of all, hydrides allow to retain large quantities of hydrogen due to the high efficiency of chemical bonds. Secondly, it is a relatively safe way of storage and transportation as compared with high-pressure gas cylinders and cryogenic systems. For instance, car hydrogen battery is a tank filled with powder-like hydride. The hydride decomposes under heating and gaseous energy carrier is released. The problem is that no material that would accumulate large quantities of hydrogen and satisfy competitive operational requirements has yet been found. However, if environmental requirements become crucial under certain conditions, the prospects of hydrogen engines are obvious.

The problem of filling also calls for an effective solution: hydriding under high pressure causes intensive heat release, which triggers a reverse reaction of decomposition. Modelling of hydrides formation is an independent problem. Let us dwell upon mathematical models of dehydriding in the context of the experimental method of thermodesorption spectroscopy (TDS). Computational experiments allow to "scan" a wide range of parameters and operating conditions of a material, and identify the limiting factors. The problems of the control of dehydriding kinetics parameters and the heating law are quite topical. We are interested in the problem for a tank with a huge number of powder particles of different sizes rather than in the "basic" problem for an individual hydride particle.

In Chapter 6, a discrete time inventory system with demands occurring according to a Bernoulli process and geometrically distributed lead time is considered, wherein a demanded item is delivered to the customers only after performing some random service. The service facility is assumed to have an infinite waiting hall. An (s, S) type ordering policy is adopted. The joint probability distribution of the number of customers in the system and the inventory level is obtained in steady state case. Some system performance measures are derived and the results are illustrated numerically.

The adaptive linearization of dynamic nonlinear systems remains, in general, as an open problem due the complexities associated to the method required to derive the linear or quasilinear model. The problem is even more difficult if the system is uncertain, that is, when the formal description of the plant is almost unknown considering that number of states is available. Chapter 7 discuses an adaptive linearization method for perturbed nonlinear uncertain systems based on the application of special artificial neural networks. The proposal is based on no-parametric identifier and its convergence is analyzed using the second method of Lyapunov. The suggested structure preserves some inherited structural properties like controllability. The scheme was tested using three different set of activation functions: sigmoid, wavelets and Chevyshev polynomials. The proposed method shows a good transient performance and the identification goals are fulfilled. A distillation column was used to show how the identifier works.

In four sections, Chapter 8 is organized as follows. In section 1 we investigate the basic problem with jumping nonlinearity

$$\begin{gathered} u''(x) + \lambda_+ u^+(x) - \lambda_- u^-(x) = 0\,, \quad x \in (0, \pi)\,, \\ u(0) = u(\pi) = 0. \end{gathered}$$

We define Fucik spectrum Σ and describe the solutions corresponding to the point $(\lambda_+, \lambda_-) \in \Sigma$. We introduce regions of type (I), (II), respectively defined by the curves of Fucik spectrum.

In section 2 we introduce some necessary notions and basic assertions. We formulate linking theorem which we use to prove the existence of the solution to our problem.

In section 3 we apply variational approach to obtain the existence results to the following nonlinear problem

$$u''(x) + \alpha u^+(x) - \beta u^-(x) + g(x, u(x)) = f(x)\,, \quad x \in (0, \pi)\,,$$
$$u(0) = u(\pi) = 0$$

where the point (α, β) falls in regions of type (I), (II), respectively. For (α, β) in region of type (II) we define the right hand side f such that the equation is not solvable and find a set of f for which we get solutions.

The last section 4 deals with the damping differential equation

$$u''(x) + cu'(x) + \alpha u^+(x) - \beta u^-(x) + g(x, u(x)) = f(x)\,, \quad x \in (0, \pi)\,,$$
$$u(0) = u(\pi) = 0$$

where $c \neq 0$.

The Quantum Information Theory is a reach source of fascinating problems in Linear and Multilinear Algebra. In Chapter 9 we shall discuss one of such problems, namely the Distillation Problem.

Let ρ_k^W, $k = 1, 2, \ldots, m$, be the critical Werner state in a bipartite $d_k \times d_k$ quantum system, i.e., the one that separates the 1-distillable Werner states from those that are 1-indistillable. We propose a new conjecture (GDC) asserting that the tensor product of ρ_k^W is 1-indistillable. This is much stronger than the familiar conjecture saying that a single critical Werner state is indistillable. We prove that GDC is true for arbitrary m provided that $d_k > 2$ for at most one index k. We reformulate GDC as an intriguing inequality for four arbitrary complex hypermatrices of type $d_1 \times \cdots \times d_m$. This hypermatrix inequality is just the special case $n = 2$ of a more general conjecture (CBS conjecture) for $2n$ arbitrary complex hypermatrices of the same type. Surprisingly, the case $n = 1$ turns out to be quite interesting as it provides hypermatrix generalization of the classical Lagrange identity. We also formulate the integral version of the CBS conjecture and derive the integral version of the hypermatrix Lagrange identity.

In applications, it turns out that the matrices one encounters typically have certain properties. For example, such matrices are almost always invertible. This phenomenon may be explained by the fact that the set of singular matrices, being of lower dimension, forms a set of measure zero. This is the coarsest way to obtain statement about properties of typical matrices.

In some case it is possible to refine such statements. In particular, if the matrices are defined over a compact field or ring then the ring Mat_n of all matrices carries a unique normalized Haar measure, or in other words, a natural probability measure. Hence, it is possible to define and study random matrices. Important matrix subrings like SL_n, GL_n, O_n, U_nm and $Sp_{n,m}$ carry similar probability measures.

For matrices over Z, no such Haar measure exists. However, it is possible to compute the probabilities in every localisation. It is tempting to define global probabilities as the product over all local probabilities. Unfortunately, this method will in general not yield a probability measure.

In Chapter 10, we will study in which situations the local reductions induce a probability measure for integral matrices, thereby answering the question what properties of integral matrices are susceptible to studies by means of probability theory.

New definitions of determinant functionals (the column and row determinants) over the quaternion division algebra are given in Chapter 11. We study their properties and relations with other well-known noncommutative determinants (Study, Moore, Diedonne, Chen) and the quasideterminants of Gelfand-Retakh. We introduce a definition of a determinant of a Hermitian matrix and a double determinant and their properties are investigated. We build the theory of invertibility of a square matrix over the quaternion division algebra relying on the introduced determinants by analogy with the classical theory in the complex case. Within the framework of the theory of the column and row determinants we obtain a determinantal representation of the inverse matrix over the quaternion algebra by analogs of the classical adjoint matrix and Cramer's rule for right and left systems of linear equations. We consider some left, right and two-sided matrix equations over the quaternion algebra and solve them by the Cramer rule as well. We investigate the problem of eigenvalues of a quaternion matrix and it's singular value decomposition. Determinantal representation of the Moore-Penrose inverse is extended to a matrix over the quaternion skew field within the framework of a theory of the column and row determinants. Using the obtained analogs of the adjoint matrix, we get Cramer's rules for the least squares solutions of left and right systems of quaternionic linear equations. As a consequence we obtain the recent results for the Moore-Penrose inverse and the least squares solution in the complex case.

Brood sorting, observed in *leptothorax unifasciatus* ant colonies, is a major example of social insects ability to solve problems at the collective level. Two processes characterize this phenomenon: a process of aggregation of all brood items in a single cluster, coupled with a process of segregation of items in concentric annuli, each containing items of different type, and ordered such a way that the smallest items are at the center and the largest at the periphery. This phenomenon has been a part of what triggered a lot of studies about swarm intelligence. Nevertheless, there is still a lot to understand about that phenomenon. We propose a detailed mathematic analysis of this entire process and that leads to understand how and why swarm intelligence may occur. Chapter 12 includes "tutorial" about the mathematic tools we used in order to show how possible and useful the theoretical analysis of some swarm models may be.

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Chapter 11

THE THEORY OF THE COLUMN AND ROW DETERMINANTS IN A QUATERNION LINEAR ALGEBRA

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Abstract

New definition of determinant functionals (the column and row determinants) over the quaternion division algebra are given in this chapter. We study their properties and relations with other well-known noncommutative determinants (Study, Moore, Diedonne, Chen) and the quasideterminants of Gelfand-Retakh. We introduce a definition of a determinant of a Hermitian matrix and a double determinant and their properties are investigated. We build the theory of invertibility of a square matrix over the quaternion division algebra relying on the introduced determinants by analogy with the classical theory in the complex case. Within the framework of the theory of the column and row determinants we obtain a determinantal representation of the inverse matrix over the quaternion algebra by analogs of the classical adjoint matrix and Cramer's rule for right and left systems of linear equations. We consider some left, right and two-sided matrix equations over the quaternion algebra and solve them by the Cramer rule as well. We investigate the problem of eigenvalues of a quaternion matrix and it's singular value decomposition. Determinantal representation of the Moore-Penrose inverse is extended to a matrix over the quaternion skew field within the framework of a theory of the column and row determinants. Using the obtained analogs of the adjoint matrix, we get Cramer's rules for the least squares solutions of left and right systems of quaternionic linear equations. As a consequence we obtain the recent results for the Moore-Penrose inverse and the least squares solution in the complex case.

Keywords: quaternion skew field noncommutative determinant, inverse matrix, quaternionic system of linear equation, Cramer's rule.

MSC: 15A33, 15A15, 15A24.

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1. Introduction

Linear algebra has accumulated a rich collection of different methods. At the transition from linear algebra over a field to linear algebra over a noncommutative ring, we want to save as many tools as we regularly use in linear algebra over a field. At the beginning of XX century, soon after the creation of Hamilton quaternion algebra, mathematics sought answer how looks algebra with noncommutative multiplication. In particular, since that time there is a problem of a determinant of matrices with noncommutative entries (which are also defined as noncommutative determinants). There are several versions of the definition noncommutative determinants. But any of the previous noncommutative determinants has not fully retained those properties which it owned for matrices over a field. In particulary, determinants of matrices over a field are multiplicative. But in [10] it is proved that there no exists an extension of the definition of determinants of real matrices to quaternion matrices, such that the multiplication theorem holds. Therefore, finding a solution to the problem of noncommutative determinants is yet continued. The theory of noncommutative determinants can be divided into three approaches.

Let $\mathrm{M}\left(n, \mathbf{R}\right)$ be the ring of $n \times n$ matrices with entries in a ring $\mathbf{R}$. The first approach [1, 6, 9] to defining the determinant of a matrix in $\mathrm{M}\left(n, \mathbf{R}\right)$ is as follows.

Definition 1.1. *Let a functional* $\mathrm{d} : \mathrm{M}\left(n, \mathbf{R}\right) \to \mathbf{R}$ *satisfy the following axioms.*

> **Axiom 1** $\mathrm{d}\left(\mathbf{A}\right) = 0$ *if and only if the matrix* $\mathbf{A}$ *is singular.*
>
> **Axiom 2** $\mathrm{d}\left(\mathbf{A} \cdot \mathbf{B}\right) = \mathrm{d}\left(\mathbf{A}\right) \cdot \mathrm{d}\left(\mathbf{B}\right)$ *for* $\forall \mathbf{B} \in \mathrm{M}\left(n, \mathbf{R}\right)$.
>
> **Axiom 3** *If the matrix* $\mathbf{A}'$ *is obtained from* $\mathbf{A}$ *by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then* $\mathrm{d}\left(\mathbf{A}\right)' = \mathrm{d}\left(\mathbf{A}\right)$.

Then a value of the functional d *is called the determinant of the matrix* $\mathbf{A} \in \mathrm{M}\left(n, \mathbf{R}\right)$.

If a determinant functional satisfies Axioms 1, 2, 3, then it takes on a value in a commutative subset of the ring. It is proved in [1]. Examples of such determinant are the determinants of Study and Diedonné.

The most famous and widely used noncommutative determinant is the Diedonné determinant. It was defined for matrices over a division ring $\mathbf{R}$ by Diedonné in 1943 [7]. His idea was to consider determinants with values in $\mathbf{R}^* \setminus [\mathbf{R}^*, \mathbf{R}^*]$ where $\mathbf{R}^*$ is the monoid of invertible elements in $\mathbf{R}$. The properties of Diedonné determinants are close to those of commutative ones, but, evidently, Diedonné determinants cannot be used for solving systems of linear equations. A determinantal representation of an inverse matrix by such determinants is impossible as well. These are just some reasons which forces to define determinant functionals unsatisfying all above-stated axioms. However Axiom 1 is considered [9] indispensable for the utility of the notion of a determinant.

In another way of looking a noncommutative determinant is defined as a rational function from entries. Herein I. M. Gelfand and V. S. Retah have reached the greatest success by the theory of quasideterminants [14, 15, 16]. An arbitrary $n \times n$ matrix over a skew field is associated with an $n \times n$ matrix whose entries are quasideterminants. The quasideterminant is not an analog of the commutative determinant but rather of a ratio of the determinant of an $n \times n$-matrix to the determinant of an $(n-1) \times (n-1)$-submatrix.

Definition 1.2. *Let I, J be two finite sets of the same cardinality n. Let $\mathbf{A} = (a_{ij})$, $i \in I$, $j \in J$ be a matrix over ring $\mathbf{R}$. For $i \in I$, $j \in J$ the (i, j)th quasideterminant $\mid \mathbf{A} \mid_{i,j}$ of $\mathbf{A} \in \mathrm{M}(n, \mathbf{R})$ is defined by the formula*

$$\mid \mathbf{A} \mid_{ij} = b_{ji}^{-1} \tag{1}$$

where $\mathbf{B} = \mathbf{A}^{-1} = (b_{ij})$.

There is an equivalent definition which is obtained by the following recurrence relations.

Definition 1.3. *If $n = 1$ so that $I = i$, $J = j$, then $\mid \mathbf{A} \mid_{ij} = a_{ij}$.*
Let $n \geq 2$ and let $\mathbf{A}^{ij}$ be the $(n-1) \times (n-1)$-matrix obtained from $\mathbf{A}$ by deleting the ith row and the jth column. Then

$$\mid \mathbf{A} \mid_{ij} = a_{ij} - \sum x_{ip} (\mid \mathbf{A}^{ij} \mid_{qp})^{-1} x_{qj}$$

Here the sum is taken over $p \in I \setminus i$, $q \in J \setminus j$.

Since quasideterminants can not be expanded by cofactors along an arbitrary row or column, an inverse matrix is not represented by the adjoint classical matrix. Despite this, quasideterminants are now widely used and naturally that means one can solve systems of linear equations using quasideterminants.

For left system of linear equations

$$\mathbf{A} \cdot \mathbf{x} = \xi,$$

where $\mathbf{A} \in \mathrm{M}(n, \mathbf{R})$ is a matrix coefficients, $\xi = (\xi_1, \ldots, \xi_n)^T$ is the known column, we have

$$x_i = \sum_{j=1}^{n} |\mathbf{A}|_{ji}^{-1} \xi_j,$$

and the analog of Cramer's rule

$$|\mathbf{A}|_{ij} x_j = |\mathbf{A}_j(\xi)|_{ij},$$

where $\mathbf{A}_l(\xi)$ is obtained from $\mathbf{A}$ by replacing the lth column by ξ.

At last, at the third approach a noncommutative determinant is defined as the alternating sum of $n!$ products of entries of a matrix but by specifying a certain ordering of coefficients in each term. E. H. Moore was the first who achieved the fulfillment of the main Axiom 1 by such definition of a noncommutative determinant. This is done not for all square matrices over a ring but rather only Hermitian matrices. Moore's theory of noncommutative determinants was introduced in [23]. Later, Dyson gave some natural generalizations and described the theory in more modern terms [9].

Moore's determinant of a Hermitian matrix $\mathbf{A} = (a_{ij})_{n \times n}$ (i.e. $a_{ij} = \overline{a_{ji}}$) over a ring $\mathbf{R}$ with an involution is introduced by induction on n in the following way ([9]).

Definition 1.4. *Denote by $\mathbf{A}(i \to j)$ the matrix obtained from Hermitian $\mathbf{A} \in \mathrm{M}(n, \mathbf{R})$ by replacing its jth column with the ith column, and then by deleting both the ith row and column. Moore's determinant is defined by the formula*

$$\mathrm{Mdet}\mathbf{A} = \begin{cases} a_{11}, & n = 1 \\ \sum\limits_{j=1}^{n} \varepsilon_{ij} a_{ij} \mathrm{Mdet}\left(\mathbf{A}(i \to j)\right), & n > 1 \end{cases} \tag{2}$$

where $\varepsilon_{kj} = \begin{cases} 1, & i = j \\ -1, & i \neq j \end{cases}$.

Another definition of this determinant is represented in [1] in terms of permutations:

$$\mathrm{Mdet}\,\mathbf{A} = \sum_{\sigma \in S_n} |\sigma| a_{n_{11} n_{12}} \cdot \ldots \cdot a_{n_{1l_1} n_{11}} \cdot a_{n_{21} n_{22}} \cdot \ldots \cdot a_{n_{rl_1} n_{r1}}.$$

The disjoint cycle representation of the permutation $\sigma \in S_n$ is written in the normal form,

$$\sigma = (n_{11} \ldots n_{1l_1})(n_{21} \ldots n_{2l_2}) \ldots (n_{r1} \ldots n_{rl_r}),$$

where, for each $i = 1, ..., r$, we have $n_{i1} < n_{im}$ for all $m > 1$, and

$$n_{11} > n_{21} > ... > n_{r1}.$$

However there was no extension of the definition of the Moore determinant to arbitrary square matrices. F. J. Dyson has emphasized this point in [9]. Longxuan Chen has offered the following decision of this problem in [4, 5]. He has defined the determinant of an arbitrary square matrix $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbf{H})$ over the quaternion skew field $\mathbf{H}$ as follows.

$$\det \mathbf{A} = \sum_{\sigma \in S_n} \varepsilon(\sigma)\, a_{n_1 i_2} \cdot a_{i_2 i_3} \ldots \cdot a_{i_s n_1} \cdot \ldots \cdot a_{n_r k_2} \cdot \ldots \cdot a_{k_l n_r},$$
$$\sigma = (n_1 i_2 \ldots i_s) \ldots (n_r k_2 \ldots k_l),$$
$$n_1 > i_2, i_3, \ldots, i_s; \ldots; n_r > k_2, k_3, \ldots, k_l,$$
$$n = n_1 > n_2 > \ldots > n_r \geq 1.$$

L. Chen has obtained a determinantal representation of an inverse matrix over the quaternion skew field even though the determinant does not satisfy Axiom 1. However this determinant also can not be expanded by cofactors along an arbitrary row or column with the exception of the nth row. Therefore he has not obtained the classical adjoint matrix or its analog as well.

He defined $\|\mathbf{A}\| := \det(\mathbf{A}^* \mathbf{A})$ as the double determinant and obtained the following determinantal representation of an inverse matrix.

Theorem 1.1. *If $\|\mathbf{A}\| := \det(\mathbf{A}^* \mathbf{A}) \neq 0$ for $\mathbf{A} = (\alpha_1, \ldots, \alpha_m)$ over $\mathbf{H}$, then exists its inverse $\mathbf{A}^{-1} = (b_{jk})$, where*

$$\overline{b_{jk}} = \frac{1}{\|\mathbf{A}\|} \omega_{kj}, \quad (j, k = 1, 2, \ldots, n),$$

where

$$\omega_{kj} = \det\left(\alpha_1 \ldots \alpha_{j-1} \alpha_n \alpha_{j+1} \ldots \alpha_{n-1} \delta_k\right)^* \left(\alpha_1 \ldots \alpha_{j-1} \alpha_n \alpha_{j+1} \ldots \alpha_{n-1} \alpha_j\right).$$

Here α_i is the ith column of $\mathbf{A}$, δ_k is the n-dimension column with 1 in the kth row and 0 in others.

If $\|\mathbf{A}\| \neq 0$, then a solution of a right system of linear equations $\sum_{j=1}^{n} \alpha_j x_j = \beta$ over $\mathbf{H}$ is represented by the next formula, defined as Cramer's formula,

$$x_j = \|\mathbf{A}\|^{-1}\overline{\mathbf{D}_j},$$

for all $j = \overline{1,n}$, where

$$\mathbf{D}_j = \det \begin{pmatrix} \alpha_1^* \\ \vdots \\ \alpha_{j-1}^* \\ \alpha_n^* \\ \alpha_{j+1}^* \\ \vdots \\ \alpha_{n-1}^* \\ \beta^* \end{pmatrix} \begin{pmatrix} \alpha_1 & \ldots & \alpha_{j-1} & \alpha_n & \alpha_{j+1} & \ldots & \alpha_{n-1} & \alpha_j \end{pmatrix}.$$

Here α_i is the ith column of $\mathbf{A}$, α_i^* is the ith row of $\mathbf{A}^*$, and β^* is the n-dimension row vector conjugated with β.

In this chapter we consider the theory of the row and column determinants over the quaternion algebra. The chapter is organized as follows. In Section 2 we consider the main provisions of the quaternion algebra.

In Section 3 definitions of the row and column determinants are given and their properties of an arbitrary quadratic matrix over the quaternion algebra (including the lemmas enable expend their by cofactors) are described.

In Section 4 we introduce the determinant of a Hermitian matrix, which coincide with the Moore determinant.

In Section 5 we establish the properties of the row and column determinants of a Hermitian matrix and its diagonalization by unimodular matrices in Section 6.

In Section 7 we gives the determinantal representation of an inverse of a Hermitian matrix.

In Section 8 we obtain the properties of the left and right corresponding Hermitian matrices.

In Section 9 we set the criterion of the corresponding Hermitian matrices and introduce the rank of a Hermitian matrix by principal minors.

In Section 10 we define the double determinant in within the framework of the theory of the row and column determinants over the quaternion algebra and its properties are given.

In Section 11 we obtain the determinantal representations of an inverse matrix by the analogs of the classical adjoint matrix.

In Section 12 we establish relations between noncommutative determinants (including the quasideterminants) and the row and column determinants.

We get Cramer's rule for left and right system of linear equations in Section 13 and some matrix equations over the quaternion algebra in Section 14. In Section 15 we gives an example of solving of some matrix equation by Cramer's rule.

In Section 15 we investigate the problem of eigenvalues of a quaternion matrix and it's singular value decomposition, and introduce the Moore-Penrose inverse of a quaternion matrix.

These results obtained in Section 15 lead us to the determinantal representation of the Moore-Penrose inverse (Section 16) and to Cramer's rule for a least squares solution of quaternion system linear equations (Section 17). In Section 18 we gives an example of finding a least squares solution of some quaternion system linear equations by Cramer's rule.

Facts set forth in Sections 2-13 are published in [17], in Sections 14-15 are published in [18] and in Sections 16-18 are published in [19, 20].

2. Quaternion Algebra

The row and column determinants are defined for quadratic matrices over a quaternion algebra $\mathbb{H}$. A quaternion algebra $\mathbb{H}(a,b)$ over a field $\mathbb{F}$ is a central simple algebra over $\mathbb{F}$ that is a four-dimensional vector space over the $\mathbb{F}$. A quaternion algebra $\mathbb{H}(a,b)$ over a field $\mathbb{F}$ with basis $\{1,i,j,k\}$ and the following multiplication rules:

$$\begin{aligned} i^2 &= a, \\ j^2 &= b, \\ ij &= k, \\ ji &= -k. \end{aligned}$$

A quaternion algebra $\mathbb{H}(a,b)$ over a field $\mathbb{F}$ is denoted $(\frac{\alpha,\beta}{\mathbb{F}})$ as well. To every quaternion algebra $\mathbb{H}(a,b)$, one can associate a quadratic form n (called the norm form) on $\mathbb{H}$ such that $\mathrm{n}(xy) = \mathrm{n}(x)\mathrm{n}(y)$ for all x and y in $\mathbb{H}$. A linear mapping $x \to \overline{x} = \mathrm{t}(x) - x$ is also defined on $\mathbb{H}$. It is an involution, i.e. $\overline{\overline{x}} = x$, $\overline{x+y} = \overline{x} + \overline{y}$ and $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$ An element $\overline{x}$ is called the conjugate of $x \in \mathbb{H}$. $\mathrm{t}(x)$ and $\mathrm{n}(x)$ are called the trace and the norm of x respectively, at that $\{\mathrm{n}(x), \mathrm{t}(x)\} \subset \mathbb{F}$ for all x in $\mathbb{H}$. They also satisfy the following conditions: $\mathrm{n}\left(\overline{x}\right) = \mathrm{n}(x)$, $\mathrm{t}\left(\overline{x}\right) = \mathrm{t}(x)$ and $\mathrm{t}\left(q \cdot p\right) = \mathrm{t}\left(p \cdot q\right)$. The last property is the rearrangement property of the trace.

Depending on the choice of $\mathbb{F}$, a and b we have only two possibilities [22]:

1.$(\frac{a,b}{\mathbb{F}})$ is a division algebra,

2. $(\frac{a,b}{\mathbb{F}})$ is isomorphic to the algebra of all 2×2 matrices with entries from $\mathbb{F}$.

(If an $\mathbb{F}$-algebra is isomorphic to a full matrix algebra over $\mathbb{F}$ we say that the algebra is split, so (2) is the split case.)

Consider some non-isomorphic quaternion algebra with division.

1. If $\mathbb{F}$ is the field of the real numbers $\mathbb{R}$, then $(\frac{a,b}{\mathbb{R}})$ is isomorphic to the Hamilton quaternion skew field **H** whenever $\alpha < 0$ and $\beta < 0$. Otherwise $(\frac{a,b}{\mathbb{R}})$ is split.

2. If $\mathbb{F}$ is the field of the real numbers $\mathbb{Q}$, then there exist infinitely many non-isomorphic division quaternion algebras $(\frac{a,b}{\mathbb{Q}})$.

3. Let $\mathbb{Q}_p$ is $p$adic field, where p is a prime. For each prime p there is a unique quaternion division algebra over $\mathbb{Q}_p$.

3. Definitions and Basic Properties of the Column and Row Determinants

To introduce the row and column determinants, we need the following definitions in the theory of permutations.

Definition 3.1. *Suppose S_n is the symmetric group on the set $I_n = \{1, \dots, n\}$. If two-line notation of a permutation $\sigma \in S_n$ corresponds to its some cycle notation, then we say that the permutation $\sigma \in S_n$ forms the direct product of disjoint cycles, i.e.*

$$\sigma = \begin{pmatrix} n_{11} & n_{12} & \dots & n_{1l_1} & \dots & n_{r1} & n_{r2} & \dots & n_{rl_r} \\ n_{12} & n_{13} & \dots & n_{11} & \dots & n_{r2} & n_{r3} & \dots & n_{r1} \end{pmatrix}. \tag{3}$$

Definition 3.2. *If cycle notation of $\sigma \in S_n$ is written as the upper row of its corresponding two-line notation, then it is called the left-ordered cycle notation of the permutation $\sigma \in S_n$. This means that if two-line notation of $\sigma \in S_n$ by the direct product of disjoint cycles has the form (3), then the left-ordered cycle notation is represented by*

$$\sigma = (n_{11}n_{12}\dots n_{1l_1})\,(n_{21}n_{22}\dots n_{2l_2})\dots(n_{r1}n_{r2}\dots n_{rl_r}).$$

We use the term ”left-ordered”, because each cycle is started from some x of I_n on the left. Then we obtain the sequence $(x\,\sigma(x)\,\sigma(\sigma(x))...)$ of successive images under σ (ordered from left to right), until the image would be x.

Definition 3.3. *If cycle notation of $\sigma \in S_n$ is written as the lower row of its corresponding two-line notation, then it is called the right-ordered cycle notation of the permutation $\sigma \in S_n$. This means that if two-line notation of $\sigma \in S_n$ by the direct product of disjoint cycles has the form (3), then the right-ordered cycle notation is represented by*

$$\sigma = (n_{12}\dots n_{1l_1}n_{11})\,(n_{22}\dots n_{2l_2}n_{21})\dots(n_{r2}\dots n_{rl_r}n_{r1}).$$

We use the term ”right-ordered”, because each cycle is started from some x of I_n on the right. Then we obtain the sequence $(...\sigma^{-1}(\sigma^{-1}(x))\ \sigma^{-1}(x)\ x)$ of successive images under σ^{-1} (ordered from right to left), until the image would be x.

Definition 3.4. *The ith row determinant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbf{H})$ is defined as the alternative sum of $n!$ monomials compounded from entries of $\mathbf{A}$ such that the index permutation of each monomials forms the direct product of disjoint cycles. If the permutation is even, then the monomial has a sign ”$+$”. If the permutation is odd, then the monomial has a sign ”$-$”. That is*

$$\mathrm{rdet}_i\mathbf{A} = \sum_{\sigma\in S_n}(-1)^{n-r}\,a_{i\,i_{k_1}}a_{i_{k_1}i_{k_1+1}}\dots a_{i_{k_1+l_1}i}\dots a_{i_{k_r}i_{k_r+1}}\dots a_{i_{k_r+l_r}i_{k_r}},$$

where S_n is the symmetric group on the set I_n. Left-ordered cycle notation of the permutation σ is written as follows

$$\sigma = (i\,i_{k_1}i_{k_1+1}\dots i_{k_1+l_1})\,(i_{k_2}i_{k_2+1}\dots i_{k_2+l_2})\dots(i_{k_r}i_{k_r+1}\dots i_{k_r+l_r}).$$

Here the index i starts the first cycle from the left and other cycles satisfy the following conditions

$$i_{k_2} < i_{k_3} < \ldots < i_{k_r}, \quad i_{k_t} < i_{k_t+s}.$$

for all $t = \overline{2,r}$ *and* $s = \overline{1,l_t}$.

Let $\mathbf{a}_{.j}$ be the jth column and $\mathbf{a}_{i.}$ be the ith row of a matrix $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$. Denote by $\mathbf{A}_{.j}\,(\mathbf{b})$ a matrix obtained from $\mathbf{A}$ by replacing its jth column with the column $\mathbf{b}$, and by $\mathbf{A}_{i.}\,(\mathbf{b})$ a matrix obtained from $\mathbf{A}$ by replacing its ith row with the row $\mathbf{b}$. Denote by $\mathbf{A}^{ij}$ a submatrix of $\mathbf{A}$ obtained by deleting both the ith row and the jth column.

The next lemma enables us to expand $\mathrm{rdet}_i\,\mathbf{A}$ by cofactors along the i-th row for all $i = \overline{1,n}$. The calculation of the row determinant of a $n \times n$ matrix is reduced to the calculation of the row determinant of a lower dimension matrix.

Lemma 3.1. *Let* R_{ij} *be the right* ij*th cofactor of* $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$*, that is* $\mathrm{rdet}_i\,\mathbf{A} = \sum\limits_{j=1}^{n} a_{ij} \cdot R_{ij}$ *for all* $i = \overline{1,n}$*. Then*

$$R_{ij} = \begin{cases} -\mathrm{rdet}_j\,\mathbf{A}^{ii}_{.j}\,(\mathbf{a}_{.\,i})\,, & i \neq j, \\ \mathrm{rdet}_k\,\mathbf{A}^{ii}, & i = j, \end{cases}$$

where $\mathbf{A}^{ii}_{.j}\,(\mathbf{a}_{.\,i})$ *is obtained from* $\mathbf{A}$ *by replacing the* j*th column with the* i*th column, and then by deleting both the* i*th row and column;* $k = \min\{I_n \setminus \{i\}\}$.

Proof. At first we prove that $R_{ii} = \mathrm{rdet}_k\,\mathbf{A}^{ii}$, where $k = \min\{I_n \setminus \{i\}\}$.

If $i = 1$, then $\mathrm{rdet}_1\,\mathbf{A} = a_{11} \cdot R_{11} + a_{12} \cdot R_{12} + \ldots + a_{1n} \cdot R_{1n}$. Consider the monomials of $\mathrm{rdet}_1\,\mathbf{A}$ such that begin with a_{11} from the left:

$$a_{11} \cdot R_{11} = \sum_{\tilde{\sigma} \in S_n} (-1)^{n-r}\, a_{11} a_{2\,i_{k_2}} \ldots a_{i_{k_2+l_2}\,2} \ldots a_{i_{k_r}\,i_{k_r+1}} \ldots a_{i_{k_r+l_r}\,i_{k_r}},$$

where $\tilde{\sigma} = (1)\,(2\,i_{k_2} \ldots i_{k_2+l_2}) \ldots (i_{k_r} i_{k_r+1} \ldots i_{k_r+l_r})$. By factoring the common left-side factor a_{11}, we obtain

$$a_{11} R_{11} = a_{11} \sum_{\tilde{\sigma}_1 \in S_{n-1}} (-1)^{n-1-(r-1)}\, a_{2\,i_{k_2}} \ldots a_{i_{k_2+l_2} 2} \ldots a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}},$$

where $\tilde{\sigma}_1 = (2\,i_{k_2} \ldots i_{k_2+l_2}) \ldots (i_{k_r} i_{k_r+1} \ldots i_{k_r+l_r})$. Here S_{n-1} is the symmetric group on $I_n \setminus \{1\}$. The numbers of the disjoint cycles and the coefficients of every monomial of R_{11} decrease by one. Each monomial of R_{11} begins on the left with some entry of the second row of $\mathbf{A}$. There are no entries of the first row and column of $\mathbf{A}$ among its coefficients. Thus, we have

$$R_{11} = \sum_{\tilde{\sigma}_1 \in S_{n-1}} (-1)^{n-1-(r-1)}\, a_{2\,i_{k_2}} \ldots a_{i_{k_2+l_2} 2} \ldots a_{i_{k_r+l_r} i_{k_r}} = \mathrm{rdet}_2 \mathbf{A}^{11}. \tag{4}$$

If now $i \neq 1$, then

$$\mathrm{rdet}_i\,\mathbf{A} = a_{i1} \cdot R_{i1} + a_{i2} \cdot R_{i2} + \ldots + a_{in} \cdot R_{in} \tag{5}$$

Consider monomials of $\mathrm{rdet}_i\,\mathbf{A}$ such that begin with $a_{i\,i}$ from the left:

$$a_{i\,i}\cdot R_{i\,i}=\sum_{\widehat{\sigma}\in S_n}(-1)^{n-r}\,a_{i\,i}a_{1\,i_{k_2}}\ldots a_{i_{k_2+l_2}1}\ldots a_{i_{k_r}i_{k_r+1}}\ldots a_{i_{k_r+l_r}i_{k_r}},$$

where $\widehat{\sigma}=(i)\,(1\,i_{k_2}\ldots i_{k_2+l_2})\ldots(i_{k_r}i_{k_r+1}\ldots i_{k_r+l_r})$. By factoring the common left-side factor $a_{i\,i}$, we get

$$a_{i\,i}\cdot R_{i\,i}=a_{i\,i}\cdot\sum_{\widehat{\sigma}_1\in\widehat{S}_{n-1}}(-1)^{n-1-(r-1)}\,a_{1\,i_{k_2}}\ldots a_{i_{k_2+l_2}1}\ldots a_{i_{k_r+l_r}i_{k_r}},$$

where $\widehat{\sigma}_1=(1\,i_{k_2}\ldots i_{k_2+l_2})\ldots(i_{k_r}i_{k_r+1}\ldots i_{k_r+l_r})$. Here $\widehat{S}_{n-1}$ is the symmetric group on $I_n\setminus\{i\}$. The numbers of disjoint cycles and the coefficients of every monomial of $R_{i\,i}$ again decrease by one. Each monomial of $R_{i\,i}$ begins on the left with an entry of the first row. There are no entries of the ith row and column of $\mathbf{A}$ among its coefficients. Thus, we obtain

$$R_{i\,i}=\sum_{\widehat{\sigma}_1\in\widehat{S}_{n-1}}(-1)^{n-1-(r-1)}\,a_{1\,i_{k_2}}\ldots a_{i_{k_2+l_2}1}\ldots a_{i_{k_r+l_r}i_{k_r}}=\mathrm{rdet}_1\mathbf{A}^{i\,i}. \qquad (6)$$

Combining (4) and (6), we get $R_{i\,i}=\mathrm{rdet}_k\,\mathbf{A}^{i\,i}$, $k=\min\{I_n\setminus\{i\}\}$.

Now suppose that $i\neq j$. Consider monomials of $\mathrm{rdet}_i\,\mathbf{A}$ in (5) such that begin with $a_{i\,j}$ from the left:

$$\begin{aligned}a_{i\,j}\cdot R_{i\,j}&=\sum_{\bar{\sigma}\in S_n}(-1)^{n-r}\,a_{i\,j}\,a_{j\,i_{k_1}}\ldots a_{i_{k_1+l_1}i}\ldots a_{i_{k_r}i_{k_r+1}}\ldots a_{i_{k_r+l_r}i_{k_r}}=\\&=-a_{i\,j}\cdot\sum_{\bar{\sigma}\in S_n}(-1)^{n-r-1}\,a_{j\,i_{k_1}}\ldots a_{i_{k_1+l_1}i}\ldots a_{i_{k_r}i_{k_r+1}}\ldots a_{i_{k_r+l_r}i_{k_r}},\end{aligned}$$

where $\bar{\sigma}=(i\,j\;i_{k_1}\ldots i_{k_1+l_1})\ldots(i_{k_r}i_{k_r+1}\ldots i_{k_r+l_r})$. Denote $\tilde{a}_{i_{k_1+l_1}j}=a_{i_{k_1+l_1}i}$ for all $i_{k_1+l_1}\in I_n$. Then

$$a_{i\,j}\cdot R_{i\,j}=-a_{i\,j}\cdot\sum_{\bar{\sigma}_1\in\,\widehat{S}_{n-1}}(-1)^{n-r-1}\,a_{j\,i_{k_1}}\ldots\tilde{a}_{i_{k_1+l_1}j}\ldots a_{i_{k_r+l_r}i_{k_r}},$$

where $\bar{\sigma}_1=(j\,i_{k_1}\ldots i_{k_1+l_1})\ldots(i_{k_r}i_{k_r+1}\ldots i_{k_r+l_r})$. The permutation $\bar{\sigma}_1$ does not contain the index i in each monomial of $R_{i\,j}$. This permutation satisfies the conditions of Definition 3.4 for $\mathrm{rdet}_j\mathbf{A}^{i\,i}_{.\,j}(\mathbf{a}_{.\,i})$. The matrix $\mathbf{A}^{i\,i}_{.\,j}(\mathbf{a}_{.\,i})$ is obtained from $\mathbf{A}$ by replacing the jth column with the column i, and then by deleting both the ith row and column. That is,

$$\sum_{\bar{\sigma}_1\in\,\widehat{S}_{n-1}}(-1)^{n-r-1}\,a_{j\,i_{k_1}}\ldots\tilde{a}_{i_{k_1+l_1}\,j}\ldots a_{i_{k_r+l_r}\,i_{k_r}}=\mathrm{rdet}_j\,\mathbf{A}^{i\,i}_{.\,j}(\mathbf{a}_{.\,i})$$

Therefore, if $i\neq j$, then $R_{ij}=-\mathrm{rdet}_j\mathbf{A}^{i\,i}_{.\,j}(\mathbf{a}_{.\,i})$.■

Definition 3.5. *The jth column determinant of $\mathbf{A}\in\mathrm{M}(n,\mathbb{H})$ is defined as the alternative sum of $n!$ monomials compounded from entries of $\mathbf{A}$ such that the index permutation of* *each monomials forms the direct product of disjoint cycles. If the permutation is even, then a monomial has a sign "+". If the permutation is odd, then a monomial has a sign "−". That is*

$$\mathrm{cdet}_j\,\mathbf{A} = \sum_{\tau\in S_n} (-1)^{n-r}\, a_{j_{k_r} j_{k_r+l_r}} \ldots a_{j_{k_r+1} j_{k_r}} \ldots a_{j\, j_{k_1+l_1}} \ldots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j},$$

where S_n is the symmetric group on the set $J_n = \{1, \ldots, n\}$. Right-ordered cycle notation of the permutation $\tau \in S_n$ is written as follows:

$$\tau = (j_{k_r+l_r} \ldots j_{k_r+1} j_{k_r}) \ldots (j_{k_2+l_2} \ldots j_{k_2+1} j_{k_2})\ (j_{k_1+l_1} \ldots j_{k_1+1} j_{k_1} j)\,.$$

Here the first cycle from the right begins with the index j and other cycles satisfy the following conditions

$$j_{k_2} < j_{k_3} < \ldots < j_{k_r}, \quad j_{k_t} < j_{k_t+s},$$

for all $t = \overline{2, r}$ and $s = \overline{1, l_t}$.

Remark 3.1. *A feature of the column determinant is that it is always constructed from right to left.*

Lemma 3.2. *Let L_{ij} be the left ijth cofactor of of a matrix $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$, that is $\mathrm{cdet}_j\,\mathbf{A} = \sum_{i=1}^{n} L_{ij} \cdot a_{ij}$ for all $j = \overline{1, n}$. Then*

$$L_{ij} = \begin{cases} -\mathrm{cdet}_i\,\mathbf{A}^{jj}_{i.}(\mathbf{a}_{j.}), & i \neq j, \\ \mathrm{cdet}_k\,\mathbf{A}^{jj}, & i = j, \end{cases}$$

where $\mathbf{A}^{jj}_{i.}(\mathbf{a}_{j.})$ is obtained from $\mathbf{A}$ by replacing the ith row with the jth row, and then by deleting both the jth row and column; $k = \min\{J_n \setminus \{j\}\}$.

The proof is similar to the proof of Lemma 3.1.

Remark 3.2. *Clearly, any monomial of each row or column determinant of a square matrix corresponds to a certain monomial of another row or column determinant such that both of them consists of the same coefficients and differ only in their ordering. If the entries of an arbitrary matrix $\mathbf{A}$ are commutative, then $\mathrm{rdet}_1\,\mathbf{A} = \ldots = \mathrm{rdet}_n\mathbf{A} = \mathrm{cdet}_1\,\mathbf{A} = \ldots = \mathrm{cdet}_n\mathbf{A}$.*

Consider the basic properties of the column and row determinants of a square matrix over $\mathbb{H}$. Their proofs immediately follow from the definitions.

Theorem 3.1. *If one of the rows (columns) of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ consists of zeros only, then $\mathrm{rdet}_i\,\mathbf{A} = 0$ and $\mathrm{cdet}_i\,\mathbf{A} = 0$ for all $i = \overline{1, n}$.*

Theorem 3.2. *If the ith row of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is left-multiplied by $b \in \mathbb{H}$, then $\mathrm{rdet}_i\,\mathbf{A}_{i.}(b \cdot \mathbf{a}_{i.}) = b \cdot \mathrm{rdet}_i\,\mathbf{A}$ for all $i = \overline{1, n}$.*

Theorem 3.3. *If the jth column of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is right-multiplied by $b \in \mathbb{H}$, then $\mathrm{cdet}_j\,\mathbf{A}_{.j}(\mathbf{a}_{.j} \cdot b) = \mathrm{cdet}_j\,\mathbf{A} \cdot b$ for all $j = \overline{1, n}$.*

Theorem 3.4. *If for* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *there exists such index* $t \in I_n$ *that* $a_{tj} = b_j + c_j$ *for all* $j = \overline{1, n}$*, then for all* $i = \overline{1, n}$

$$\begin{aligned} \mathrm{rdet}_i\, \mathbf{A} &= \mathrm{rdet}_i\, \mathbf{A}_{t.}\left(\mathbf{b}\right) + \mathrm{rdet}_i\, \mathbf{A}_{t.}\left(\mathbf{c}\right), \\ \mathrm{cdet}_i\, \mathbf{A} &= \mathrm{cdet}_i\, \mathbf{A}_{t.}\left(\mathbf{b}\right) + \mathrm{cdet}_i\, \mathbf{A}_{t.}\left(\mathbf{c}\right), \end{aligned}$$

where $\mathbf{b} = (b_1, \dots, b_n)$*,* $\mathbf{c} = (c_1, \dots, c_n)$*.*

Theorem 3.5. *If for* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *there exists such index* $t \in J_n$ *such that* $a_{i\,t} = b_i + c_i$ $i = \overline{1, n}$*, then for all* $j = \overline{1, n}$

$$\begin{aligned} \mathrm{rdet}_j\, \mathbf{A} &= \mathrm{rdet}_j\, \mathbf{A}_{.\,t}\left(\mathbf{b}\right) + \mathrm{rdet}_j\, \mathbf{A}_{.\,t}\left(\mathbf{c}\right), \\ \mathrm{cdet}_j\, \mathbf{A} &= \mathrm{cdet}_j\, \mathbf{A}_{.\,t}\left(\mathbf{b}\right) + \mathrm{cdet}_j \mathbf{A}_{.\,t}\left(\mathbf{c}\right), \end{aligned}$$

where $\mathbf{b} = (b_1, \dots, b_n)^T$*,* $\mathbf{c} = (c_1, \dots, c_n)^T$*.*

Theorem 3.6. *If* $\mathbf{A}^*$ *is the Hermitian adjoint matrix of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, then* $\mathrm{rdet}_i\, \mathbf{A}^* = \overline{\mathrm{cdet}_i\, \mathbf{A}}$ *for all* $i = \overline{1, n}$*.*

Remark 3.3. *Since the column and row determinants of an arbitrary square matrix over* $\mathbb{H}$ *do not satisfy Axiom 1 but these determinants are defined by analogy to the determinant of a complex square matrix, then we can consider theirs as pre-determinants.*

4. A Determinant of a Hermitian Matrix

The following lemma is needed for the sequel.

Lemma 4.1. *Let* T_n *be the sum of all possible products of the* n *factors, each of which are either* $h_i \in \mathbb{H}$ *or* $\overline{h_i}$ *for all* $i = \overline{1, n}$*, by specifying the ordering in the terms, i.e.:*

$$T_n = h_1 \cdot h_2 \cdot \ldots \cdot h_n + \overline{h_1} \cdot h_2 \cdot \ldots \cdot h_n + \ldots + \overline{h_1} \cdot \overline{h_2} \cdot \ldots \cdot \overline{h_n}.$$

Then T_n *consists of the* 2^n *terms and* $T_n = \mathrm{t}\left(h_1\right)\ \mathrm{t}\left(h_2\right)\ \ldots\ \mathrm{t}\left(h_n\right).$

Proof. The number 2^n of terms of the sum T_n is equal to the number of ordered combinations of n unknown elements with two values.

The proof goes by induction on n.
(i) If $n = 1$, then $T_1 = \overline{h_1} + h_1 = \mathrm{t}\left(h_1\right)$.
(ii) Suppose the lemma is true for $n - 1$:

$$\begin{aligned} T_{n-1} = h_1 \cdot h_2 \cdot \ldots \cdot h_{n-1} + \overline{h_1} \cdot h_2 \cdot \ldots \cdot h_{n-1} + \ldots + \overline{h_1} \cdot \overline{h_2} \cdot \ldots \cdot \overline{h_{n-1}} = \\ = \mathrm{t}\left(h_1\right)\ \mathrm{t}\left(h_2\right)\ \ldots\ \mathrm{t}\left(h_{n-1}\right). \end{aligned}$$

(iii) Now we prove that it is valid for n.

$$T_n = h_1 \cdot h_2 \cdot \ldots \cdot h_n + \overline{h_1} \cdot h_2 \cdot \ldots \cdot h_n + \ldots + \overline{h_1} \cdot \overline{h_2} \cdot \ldots \cdot \overline{h_n}.$$

By factoring the right-side common factors either $\overline{h_n}$ or h_n respectively, we obtain

$$\begin{aligned} T_n = T_{n-1} \cdot h_n + T_{n-1} \cdot \overline{h_n} = T_{n-1} \cdot \left(h_n + \overline{h_n}\right) = T_{n-1} \cdot \mathrm{t}\left(h_n\right) = \\ = \mathrm{t}\left(h_1\right) \cdot\ \mathrm{t}\left(h_2\right) \cdot\ \ldots \cdot \mathrm{t}\left(h_{n-1}\right) \cdot\ \mathrm{t}\left(h_n\right). \blacksquare \end{aligned}$$

Theorem 4.1. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is a Hermitian matrix, then*

$$\mathrm{rdet}_1\mathbf{A} = \ldots = \mathrm{rdet}_n\mathbf{A} = \mathrm{cdet}_1\mathbf{A} = \ldots = \mathrm{cdet}_n\mathbf{A} \in \mathbb{F}.$$

Proof. At first we note that if a matrix $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is Hermitian, then we have $a_{ii} \in \mathbf{R}$ and $a_{ij} = \overline{a_{ji}}$ for all $i, j = \overline{1, n}$.

We divide the set of monomials of $\mathrm{rdet}_i\mathbf{A}$ for some $i = \overline{1, n}$ into two subsets. If indices of coefficients of monomials form permutations as products of disjoint cycles of length 1 and 2, then we include these monomials in the first subset. Other monomials belong to the second subset. If indices of coefficients form a disjoint cycle of length 1, then these coefficients are entries of the principal diagonal of the Hermitian matrix $\mathbf{A}$. Hence, they belong to $\mathbb{F}$. If indices of coefficients form a disjoint cycle of length 2, then these entries are conjugated, $a_{i_k i_{k+1}} = \overline{a_{i_{k+1} i_k}}$, and their product takes on a value in $\mathbb{F}$ as well,

$$a_{i_k i_{k+1}} \cdot a_{i_{k+1} i_k} = \overline{a_{i_{k+1} i_k}} \cdot a_{i_{k+1} i_k} = \mathrm{n}(a_{i_{k+1} i_k}) \in \mathbb{F}.$$

So, all monomials of the first subset take on values in $\mathbb{F}$.

Now we consider some monomial d of the second subset. Assume that its index permutation forms a direct product of r disjoint cycles. Denote $i_{k_1} := i$.

$$\begin{array}{l} d = (-1)^{n-r} a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i_{k_1}} a_{i_{k_2} i_{k_2+1}} \ldots a_{i_{k_2+l_2} i_{k_2}} \ldots a_{i_{k_m} i_{k_m+1}} \ldots \times \\ \times a_{i_{k_m+l_m} i_{k_m}} \ldots a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}} = (-1)^{n-r} h_1 h_2 \ldots h_m \ldots h_r, \end{array} \tag{7}$$

where $h_s = a_{i_{k_s} i_{k_s+1}} \cdot \ldots \cdot a_{i_{k_s+l_s} i_{k_s}}$ for all $s = \overline{1, r}$, and $m \in \{1, \ldots, r\}$. If $l_s = 1$, then $h_s = a_{i_{k_s} i_{k_s+1}} a_{i_{k_s+1} i_{k_s}} = \mathrm{n}(a_{i_{k_s} i_{k_s+1}}) \in \mathbb{F}$. If $l_s = 0$, then $h_s = a_{i_{k_s} i_{k_s}} \in \mathbb{F}$. If $l_s = 0$ or $l_s = 1$ for all $s = \overline{1, r}$ in (7), then we obtain a monomial of the first subset. Let there exists $s \in I_n$ such that $l_s \geq 2$. Then

$$\overline{h_s} = \overline{a_{i_{k_s} i_{k_s+1}} \ldots a_{i_{k_s+l_s} i_{k_s}}} = \overline{a_{i_{k_s+l_s} i_{k_s}}} \ldots \overline{a_{i_{k_s} i_{k_s+1}}} = a_{i_{k_s} i_{k_s+l_s}} \ldots a_{i_{k_s+1} i_{k_s}}.$$

Denote by $\sigma_s(i_{k_s}) := (i_{k_s} i_{k_s+1} \ldots i_{k_s+l_s})$ a disjoint cycle of indices of d for some $s = \overline{1, r}$. The disjoint cycle $\sigma_s(i_{k_s})$ corresponds to the factor h_s. Then $\sigma_s^{-1}(i_{k_s}) = (i_{k_s} i_{k_s+l_s} i_{k_s+1} \ldots i_{k_s+1})$ is the inverse disjoint cycle and $\sigma_s^{-1}(i_{k_s})$ corresponds to the factor $\overline{h_s}$. By Lemma 4.1 there exist another $2^p - 1$ monomials for d, (where $p = r - \rho$ and ρ is the number of disjoint cycles of length 1 and 2), such that their index permutations form the direct products of r disjoint cycles either $\sigma_s(i_{k_s})$ or $\sigma_s^{-1}(i_{k_s})$ by specifying their ordering by s from 1 to r. Their cycle notations are left-ordered according to Definition 3.4. Suppose C_1 is the sum of these $2^p - 1$ monomials and d, then by Lemma 4.1 we obtain

$$C_1 = (-1)^{n-r} \alpha \, \mathrm{t}(h_{\nu_1}) \ \ldots \ \mathrm{t}(h_{\nu_p}) \in \mathbb{F}.$$

Here $\alpha \in \mathbb{F}$ is the product of coefficients whose indices form disjoint cycles of length 1 and 2, $\nu_k \in \{1, \ldots, r\}$ for all $k = \overline{1, p}$.

Thus for an arbitrary monomial of the second subset of $\mathrm{rdet}_i\,\mathbf{A}$, we can find the 2^p monomials such that their sum takes on a value in $\mathbb{F}$. Therefore, $\mathrm{rdet}_i\,\mathbf{A} \in \mathbb{F}$.

Now we prove the equality of all row determinants of $\mathbf{A}$. Consider an arbitrary $\mathrm{rdet}_j\,\mathbf{A}$ such that $j \neq i$ for all $j = \overline{1, n}$. We divide the set of monomials of $\mathrm{rdet}_j\,\mathbf{A}$ into two subsets using the same rule as for $\mathrm{rdet}_i\, \mathbf{A}$. Monomials of the first subset are products of entries of the principal diagonal of $\mathbf{A}$ or norms of entries. Therefore they take on a value in $\mathbb{F}$ and each monomial of the first subset of $\mathrm{rdet}_i\, \mathbf{A}$ is equal to a corresponding monomial of the first subset of $\mathrm{rdet}_j\, \mathbf{A}$.

Now consider the monomial d_1 of the second subset of monomials of $\mathrm{rdet}_i\, \mathbf{A}$ consisting of coefficients that are equal to the coefficients of d but are placed in another arrangement. Consider all possibilities of the arrangement of coefficients in d_1.

(i) Suppose that the index permutation of its coefficients form a direct product of r disjoint cycles and these cycles coincide with the r disjoint cycles of d but differ by their ordering Then we have

$$d_1 = (-1)^{n-r} \alpha h_\mu \dots h_\lambda,$$

where $\{\mu, \dots, \lambda\} = \{\nu_1, \dots, \nu_p\}$. By Lemma 4.1 there exist $2^p - 1$ monomials of the second subset of $\mathrm{rdet}_j\, \mathbf{A}$ such that each of them is equal to a product of p factors either h_s or $\overline{h_s}$ for all $s \in \{\mu, \dots, \lambda\}$, multiplied by $(-1)^{n-r}\alpha$. Hence by Lemma 4.1, we obtain

$$C_2 = (-1)^{n-r} \alpha\, t(h_\mu)\ \dots\ t(h_\lambda) = (-1)^{n-r}\ \alpha\, t(h_{\nu_1}) \dots\ t(h_{\nu_p}) = C_1.$$

(ii) Now suppose that in addition to the case (i) the index j is placed inside some disjoint cycle of the index permutation of d, e.g. $j \in \{i_{k_{m+1}}, ..., i_{k_m + l_m}\}$. Denote $j = i_{k_m+q}$. Then d_1 is represented as follows:

$$\begin{array}{c} d_1 = (-1)^{n-r} a_{i_{k_m+q} i_{k_m+q+1}} \cdots \quad a_{i_{k_m+l_m} i_{k_m}}\, a_{i_{k_m} i_{k_m+1}} \cdots \times \\ \times a_{i_{k_m+q-1} i_{k_m+q}} a_{i_{k_\mu} i_{k_\mu+1}} \dots a_{i_{k_\mu+\tilde{l}_\mu} i_{k_\mu}} \dots a_{i_{k_\lambda} i_{k_\lambda+1}} \dots a_{i_{k_\lambda+l_\lambda} i_{k_\lambda}} = \\ = (-1)^{n-r} \alpha \tilde{h}_m h_\mu \dots h_\lambda, \end{array} \tag{8}$$

where $\{m, \mu, \dots, \lambda\} = \{\nu_1, \dots, \nu_p\}$. Except for $\tilde{h}_m$, each factor of d_1 in (8) corresponds to the equal factor of d in (7). By the rearrangement property of the trace, we have $t(\tilde{h}_m) = t(h_m)$. Hence by Lemma 4.1 and by analogy to the previous case, we obtain the following equality.

$$\begin{array}{c} C_2 = (-1)^{n-r} \alpha\, t(\tilde{h}_m)\, t(h_\mu)\ \dots\ t(h_\lambda) = \\ = (-1)^{n-r}\ \alpha\, t(h_{\nu_1})\ \dots\ t(h_m)\ \dots\ t(h_{\nu_p}) = C_1. \end{array}$$

(iii) If in addition to the case (i) the index i is placed inside some disjoint cycles of the index permutation of d_1, then we apply the rearrangement property of the trace to this cycle. As in the previous cases we find 2^p monomials of the second subset of $\mathrm{rdet}_j\, \mathbf{A}$ such that by Lemma 4.1 their sum is equal to the sum of the corresponding 2^p monomials of $\mathrm{rdet}_i \mathbf{A}$. Clearly, we obtain the same conclusion at association of all previous cases, then we apply twice the rearrangement property of the trace.

Thus, in any case each sum of 2^p corresponding monomials of the second subset of $\mathrm{rdet}_j\, \mathbf{A}$ is equal to the sum of 2^p monomials of $\mathrm{rdet}_i\, \mathbf{A}$. Here p is the number of disjoint cycles of length more than 2. Therefore, for all $i, j = \overline{1, n}$ we have

$$\mathrm{rdet}_i\, \mathbf{A} = \mathrm{rdet}_j\, \mathbf{A} \in \mathbb{F}.$$

Now we prove the equality $\mathrm{cdet}_i\, \mathbf{A} = \mathrm{rdet}_i\, \mathbf{A}$ for all $i = \overline{1, n}$. Again we divide the set of monomials of $\mathrm{cdet}_i\, \mathbf{A}$ into two subsets by following the same rule as for $\mathrm{rdet}_i\, \mathbf{A}$. Each monomial of the first subset of $\mathrm{cdet}_i\,\mathbf{A}$ is equal to the corresponding monomial of $\mathrm{rdet}_i\,\mathbf{A}$, since their factors are real numbers (either entries of the principal diagonal of $\mathbf{A}$ or norms of entries of $\mathbf{A}$). Consider the monomial d_2 of the second subset of monomials of $\mathrm{cdet}_i\,\mathbf{A}$ consisting of coefficients that are equal to the coefficients of d. The coefficients of d_2 are placed in the same ordering as for d but from left to right. If ρ is the number of disjoint cycles of length 1 and 2, and $p = r - \rho$, then

$$\begin{gathered} d_2 = (-1)^{n-r} a_{i_{k_r} i_{k_r+l_r}} \ldots a_{i_{k_r+1} i_{k_r}} \ldots a_{i_{k_2} i_{k_2+l_2}} \ldots a_{i_{k_2+1} i_{k_2}} \times \\ \times a_{i_{k_1} i_{k_1+l_1}} \ldots a_{i_{k_1+1} i_{k_1}} = (-1)^{n-r} \alpha\, h_{\tau_p} \ldots h_{\tau_1} \end{gathered}$$

Here α is a product of coefficients whose indices form disjoint cycles of length 1 and 2. We have for all $s = \overline{1,p}$

$$h_{\tau_s} = a_{i_{k_s} i_{k_s+l_s}} \cdot \ldots \cdot a_{i_{k_s+1} i_{k_s}} = \overline{a_{i_{k_s} i_{k_s+1}} \cdot \ldots \cdot a_{i_{k_s+l_s} i_{k_s}}}.$$

By Lemma 4.1 among monomials of the second subset of $\mathrm{cdet}_i\,\mathbf{A}$, there exist $2^p - 1$ monomials for d_2 such that each of them is equal to a product of p factors either h_{τ_s} or $\overline{h_{\tau_s}}$ for some $s = \overline{1,p}$ by specifying their right-ordering, and is multiplied by $(-1)^{n-r}\alpha$. Consider the sum C_3 of these monomials and d. Due to commutativity of real numbers and by Lemma 4.1, we get

$$\begin{gathered} C_3 = (-1)^{n-r} \alpha\, t(h_{\tau_p}) \;\ldots\; t(h_{\tau_1}) = (-1)^{n-r} \alpha\, t(\overline{h_{\nu_p}}) \;\ldots\; t(\overline{h_{\nu_1}}) = \\ = (-1)^{n-r} \alpha\, t(h_{\nu_1}) \ldots\; t(h_{\nu_p}) = C_1 \end{gathered}$$

Therefore, each sum of the 2^p corresponding monomials of the second subset of $\mathrm{cdet}_i\,\mathbf{A}$ is equal to a sum of the 2^p monomials of $\mathrm{rdet}_i\,\mathbf{A}$ and vice versa.

Thus, $\mathrm{cdet}_i\,\mathbf{A} = \mathrm{rdet}_i\,\mathbf{A} \in \mathbf{R}$ for all $i = \overline{1,n}$.■

Remark 4.1. *Since all column and row determinants of a Hermitian matrix over $\mathbb{H}$ are equal, we can define the determinant of a Hermitian matrix $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$. By definition, we put for all $i = \overline{1,n}$*

$$\det \mathbf{A} := \mathrm{rdet}_i\,\mathbf{A} = \mathrm{cdet}_i\,\mathbf{A}.$$

Remark 4.2. *By Lemma 4.1 we have*

$$\det \mathbf{A} = -\sum_{\sigma \in I_n} a_{ij} \cdot \mathrm{rdet}_j\,\mathbf{A}^{ii}_{.j}\,(\mathbf{a}_{.i}) + a_{ii} \cdot \mathrm{rdet}_k\,\mathbf{A}^{ii},\; k = \min\{I_n \setminus \{i\}\}. \tag{9}$$

By comparing expressions (2) and (9) for Hermitian $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$, we conclude that the row determinant of a Hermitian matrix coincides with the Moore determinant. Hence the row and column determinants extend the Moore determinant to an arbitrary square matrix.

5. Properties of the Column and Row Determinants of a Hermitian Matrix

Theorem 5.1. *If the matrix $\mathbf{A}_{j.}\,(\mathbf{a}_{i.})$ is obtained from a Hermitian matrix $\mathbf{A} \in M\,(n, \mathbb{H})$ by replacing its jth row with the ith row, then for all $i, j = \overline{1,n}$ such that $i \neq j$ we have*

$$\mathrm{rdet}_j \mathbf{A}_{j\,.}\,(\mathbf{a}_{i\,.}) = 0.$$

Proof. We assume $n > 3$ for $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$. The case $n \leq 3$ is easily proved by a simple check. Consider some monomial d of $\mathrm{rdet}_j\, \mathbf{A}_{j\,.}(\mathbf{a}_{i\,.})$. Suppose the index permutation of its coefficients forms a direct product of r disjoint cycles, and denote $i = i_s$. Consider all possibilities of disposition of an entry of the i_sth row in the monomial d.

(i) Suppose an entry of the i_sth row is placed in d such that the index i_s opens some disjoint cycle, i.e.:

$$d = (-1)^{n-r} a_{j\, i_1} \ldots a_{i_k j}\, u_1 \ldots u_\rho\, a_{i_s i_{s+1}} \ldots a_{i_{s+m} i_s}\, v_1 \ldots v_p \tag{10}$$

Here we denote by u_τ and v_t products of coefficients whose indices form some disjoint cycles for all $\tau = \overline{1, \rho}$ and $t = \overline{1, p}$ such that $\rho + p = r - 2$ or there are no such products. For d there are the following three monomials of $\mathrm{rdet}_j\, \mathbf{A}_{j\,.}(\mathbf{a}_{i\,.})$.

$$\begin{aligned} d_1 &= (-1)^{n-r+1} a_{j\, i_{s+1}} \ldots a_{i_{s+m} i_s}\, a_{i_s i_1} \ldots a_{i_k j}\, u_1 \ldots u_\rho\, v_1 \ldots v_p, \\ d_2 &= (-1)^{n-r+1} a_{j\, i_{s+m}} \ldots a_{i_{s+1} i_s} a_{i_s\, i_1} \ldots a_{i_k j}\, u_1 \ldots u_\rho\, v_1 \ldots v_p, \\ d_3 &= (-1)^{n-r} a_{j\, i_1} \ldots a_{i_k j}\, u_1 \ldots u_\rho\, a_{i_s i_{s+m}} \ldots a_{i_{s+1} i_s}\, v_1 \ldots v_p. \end{aligned}$$

Suppose $a_{j i_1} \ldots a_{i_k j} = x$ and $a_{i_s i_{s+1}} \ldots a_{i_{s+m} i_s} = y$, then $\overline{y} = a_{i_s i_{s+m}} \ldots a_{i_{s+1}\, i_s}$. Taking into account $a_{j\, i_1} = a_{i_s i_1}$, $a_{j\, i_{s-1}} = a_{i_s i_{s-1}}$ and $a_{j\, i_{s+1}} = a_{i_s i_{s+1}}$, we consider the sum of these monomials.

$$\begin{gathered} d + d_1 + d_2 + d_3 = (-1)^{n-r}(x u_1 \ldots u_\rho\, y - y x u_1 \ldots u_\rho - \overline{y} \cdot x u_1 \ldots u_\rho + \\ + x u_1 \ldots u_\rho \overline{y}) v_1 \ldots v_p = (-1)^{n-r}(x u_1 \ldots u_\rho t(y) - t(y) x u_1 \ldots u_\rho) v_1 \ldots v_p = 0. \end{gathered}$$

Thus among the monomials of $\mathrm{rdet}_j\, \mathbf{A}_{j\,.}(\mathbf{a}_{i\,.})$ we find three monomials for d such that the sum of these monomials and d is equal to zero.

If in (10) $m = 0$ or $m = 1$, we obtain such monomials accordingly:

$$\begin{aligned} \tilde{d} &= (-1)^{n-r} a_{j\, i_1} \ldots \cdot a_{i_k j}\, u_1 \ldots u_\rho\, a_{i_s i_s}\, v_1 \ldots v_p, \\ \overset{\frown}{d} &= (-1)^{n-r} a_{j\, i_1} \ldots a_{i_k j}\, u_1 \ldots u_\rho\, a_{i_s i_{s+1}}\, a_{i_{s+1} i_s}\, v_1 \ldots v_p. \end{aligned}$$

There are the following monomials for them:

$$\begin{aligned} \tilde{d}_1 &= (-1)^{n-r+1} a_{j\, i_s}\, a_{i_s\, i_1} \ldots a_{i_k j}\, u_1 \ldots u_\rho\, v_1 \ldots v_p, \\ \overset{\frown}{d}_1 &= (-1)^{n-r+1} a_{j\, i_{s+1}}\, a_{i_{s+1}\, i_s}\, a_{i_s\, i_1} \ldots a_{i_k j}\, u_1 \ldots u_\rho\, v_1 \ldots v_p. \end{aligned}$$

Taking into account $a_{j\, i_1} = a_{i_s i_1}$, $a_{j\, i_s} = a_{i_s i_s} \in \mathbb{F}$, $a_{j\, i_{s+1}} = a_{i_s i_{s+1}}$, and $a_{i_s i_{s+1}} a_{i_{s+1} i_s} \in \mathbb{F}$, we get $\tilde{d} + \tilde{d}_1 = 0$, $\overset{\frown}{d} + \overset{\frown}{d}_1 = 0$. Hence, the sums of corresponding two monomials of $\mathrm{rdet}_j\, \mathbf{A}_{j\,.}(\mathbf{a}_{i\,.})$ are equal to zero in this case.

ii) Now suppose that the index i_s is placed in another disjoint cycle than the index j and does not open this cycle,

$$\breve{d} = (-1)^{n-r} a_{j i_1} \ldots a_{i_k j}\, u_1 \ldots u_\rho\, a_{i_q i_{q+1}} \ldots a_{i_{s-1} i_s} a_{i_s i_{s+1}} \ldots a_{i_{q-1} i_q} v_1 \ldots v_p.$$

Here we denote by u_τ and v_t products of coefficients whose indices form some disjoint cycles for all $\tau = \overline{1, \rho}$ and $t = \overline{1, p}$ such that $\rho + p = r - 2$ or there are no such products. Now for d there are the following three monomials of $\mathrm{rdet}_j\, \mathbf{A}_{j\,.}(\mathbf{a}_{\mathrm{i}\,.})$:

$$\begin{aligned} \breve{d}_1 &= (-1)^{n-r+1} a_{j\, i_{s+1}} \ldots a_{i_{q-1} i_q}\, a_{i_q i_{q+1}} \ldots a_{i_{s-1} i_s} a_{i_s i_1} \ldots a_{i_k j}\, u_1 \ldots u_\rho v_1 \ldots v_p, \\ \breve{d}_2 &= (-1)^{n-r+1} a_{j\, i_{s-1}} \ldots a_{i_{q+1} i_q}\, a_{i_q i_{q-1}} \ldots a_{i_{s+1} i_s} a_{i_s i_1} \ldots a_{i_k j}\, u_1 \ldots u_\rho v_1 \ldots v_p, \\ \breve{d}_3 &= (-1)^{n-r} a_{j\, i_1} \ldots a_{i_k j}\, u_1 \ldots u_\rho\, a_{i_q i_{q-1}} \ldots a_{i_{s+1} i_s} a_{i_s i_{s-1}} \ldots \cdot a_{i_{q+1} i_q} v_1 \ldots v_p. \end{aligned}$$

Assume that $a_{i_s i_{s+1}} \dots a_{i_{q-1} i_q} = \varphi$, $a_{i_q i_{q+1}} \dots a_{i_{s-1} i_s} = \phi$, $a_{j\, i_1} \dots a_{i_k j} = x$, $a_{i_q i_{q+1}} \dots a_{i_{s-1} i_s} a_{i_s i_{s+1}} \dots a_{i_{q-1} i_q} = y$, $a_{i_s i_{s+1}} \dots a_{i_{q-1} i_q} a_{i_q i_{q+1}} \dots a_{i_{s-1} i_s} = y_1$. Then we obtain $y = \phi\varphi$, $y_1 = \varphi\phi$, $\overline{y} = a_{i_q i_{q-1}} \dots a_{i_{s+1} i_s} a_{i_s i_{s-1}} \dots a_{i_{q+1} i_q}$, and $\overline{y_1} = a_{i_s i_{s-1}} \dots a_{i_{q+1} i_q} a_{i_q i_{q-1}} \dots a_{i_{s+1} i_s}$. Accounting for $a_{j\, i_1} = a_{i_s i_1}$, $a_{j i_{s-1}} = a_{i_s i_{s-1}}$, $a_{j i_{s+1}} = a_{i_s i_{s+1}}$, we have

$$\begin{gathered}
\breve{d} + \breve{d}_1 + \breve{d}_2 + \breve{d}_3 = \\
= (-1)^{n-r} (x u_1 \dots u_\rho y - y_1 x u_1 \dots u_\rho - \overline{y_1}\, x u_1 \dots u_\rho + x u_1 \dots u_\rho \overline{y}) \times \\
\times v_1 \dots v_p = (-1)^{n-r} (x u_1 \dots u_\rho t(y) - t(y_1) x u_1 \dots u_\rho) v_1 \dots v_p = \\
= (-1)^{n-r} (t(\phi \cdot \varphi) - t(\varphi \cdot \phi)) x u_1 \dots u_\rho v_1 \dots v_p.
\end{gathered}$$

Since by the rearrangement property of the trace $t(\phi \cdot \varphi) = t(\varphi \cdot \phi)$, then we obtain $\breve{d} + \breve{d}_1 + \breve{d}_2 + \breve{d}_3 = 0$.

(iii) If the indices i_s and j are placed in the same cycle, then we have the following monomials: d_1, $\tilde{d}_1$, $\overset{\frown}{d}_1$ or $\breve{d}_1$. As shown above, for each of them there are another one or three monomials of $\mathrm{rdet}_j\, \mathbf{A}_{j\,.}(\mathbf{a}_{i\,.})$ such that the sums of these two or four corresponding monomials are equal to zero.

We have considered all possible kinds of disposition of an entry of the i_sth row as a factor of some monomial d of $\mathrm{rdet}_j\, \mathbf{A}_{j\,.}(\mathbf{a}_{i\,.})$. In each case there exist one or three corresponding monomials for d such that the sum of the two or four monomials is equal to zero respectively. Hence, $\mathrm{rdet}_j\, \mathbf{A}_{j\,.}(\mathbf{a}_{i\,.}) = 0$. ■

Corollary 5.1. *If a Hermitian matrix* $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$ *consists two same rows (columns), then* $\det \mathbf{A} = 0$.

Proof. Suppose the ith row of $\mathbf{A}$ coincides with the jth row, i.e. $a_{ik} = a_{jk}$ for all $k \in I_n$ and $\{i, j\} \in I_n$ such that $i \neq j$. Then $\overline{a_{ik}} = \overline{a_{jk}}$ for all $k \in I_n$. Since the matrix $\mathbf{A}$ is Hermitian, we get for all $k \in I_n$ that $a_{ki} = a_{kj}$, where $\{i, j\} \in I_n$ and $i \neq j$. This means that if a Hermitian matrix has two same rows, then it has two same corresponding columns as well. The matrix $\mathbf{A}$ may be represented as $\mathbf{A}_{j.}\,(\mathbf{a}_{i.})$, where the matrix $\mathbf{A}_{j.}\,(\mathbf{a}_{i.})$ is obtained from $\mathbf{A}$ by replacing the jth row with the ith row. Then by Theorem 5.1, we have

$$\det \mathbf{A} = \mathrm{rdet}_i \mathbf{A} = \mathrm{rdet}_i \mathbf{A}_{j.}\,(\mathbf{a}_{i.}) = 0. ■$$

The next theorem is proved in a similar way to Theorem 5.1.

Theorem 5.2. *If the matrix* $\mathbf{A}_{.\,i}\,(\mathbf{a}_{.j})$ *is obtained from a Hermitian matrix* $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$ *by replacing of its* i*th column with the* j*th column, then* $\mathrm{cdet}_i\, \mathbf{A}_{.\,i}(\mathbf{a}_{.j}) = 0$ *for all* $i, j = \overline{1, n}$ *such that* $i \neq j$.

Theorem 5.3. *If the matrix* $\mathbf{A}_{i.}\,(b \cdot \mathbf{a}_{j\,.})$ *is obtained from a Hermitian matrix* $\mathbf{A} \in M\,(n, \mathbb{H})$ *by replacing of its* i*th row with the* j*th row multiplied by* $b \in \mathbb{H}$ *on the left, then* $\mathrm{rdet}_i\, \mathbf{A}_{i.}\,(b \cdot \mathbf{a}_{j\,.}) = 0$ *for all* $i, j = \overline{1, n}$ *such that* $i \neq j$.

The proof follows immediately from Theorems 3.2 and 5.1.

Theorem 5.4. *If the matrix* $\mathbf{A}_{.\,j}\,(\mathbf{a}_{.\,i} \cdot b)$ *is obtained from a Hermitian matrix* $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$ *by replacing of its* j*th column with the* i*th column right-multiplied by* $b \in \mathbb{H}$*, then* $\mathrm{cdet}_j\, \mathbf{A}_{.\,j}\,(\mathbf{a}_{.\,i} \cdot b) = 0$ *for all* $i, j = \overline{1, n}$ *such that* $i \neq j$.

The proof follows immediately from Theorems 3.3 and 5.2.

Theorem 5.5. *If the matrix* $\mathbf{A}_{.j}\left(\mathbf{a}_{.i}\cdot b\right)$ *is obtained from a Hermitian matrix* $\mathbf{A}\in \mathrm{M}\left(n,\mathbb{H}\right)$ *by replacing of its* j*th column with the* i*th column right-multiplied by* $b\in\mathbb{H}$*, then* $\mathrm{rdet}_j\,\mathbf{A}_{.j}\left(\mathbf{a}_{.i}\cdot b\right)=0$ *for all* $i,j=\overline{1,n}$ *such that* $i\neq j$*.*

Proof. We assume $n>3$ for $\mathbf{A}\in\mathrm{M}\left(n,\mathbb{H}\right)$. The case $n\leq 3$ is easily proved by a simple check. Consider some monomial d of $\mathrm{rdet}_j\,\mathbf{A}_{.j}\left(\mathbf{a}_{.i}\cdot b\right)$ for all $i,j=\overline{1,n}$ such that $i\neq j$.

Suppose the index permutation of its coefficients forms a direct product of r disjoint cycles, and denote $i=i_s$. Consider all possibilities of disposition of an entry of the i_sth row in the monomial d.

(i) Suppose an entry of the i_sth row is placed in d such that the index i_s opens some disjoint cycle, i.e.:

$$d=(-1)^{n-r}a_{j\,i_1}\dots a_{i_kj}\,b\,u_1\dots u_\rho\,a_{i_si_{s+1}}\dots a_{i_{s+m}i_s}v_1\dots v_p,\tag{11}$$

Here we denote by u_τ and v_t products of coefficients whose indices form some disjoint cycles for all $\tau=\overline{1,\rho}$ and $t=\overline{1,p}$ such that $\rho+p=r-2$ or there are no such products. For d there are the following three monomials of $\mathrm{rdet}_j\,\mathbf{A}_{.j}\left(\mathbf{a}_{.i}\cdot b\right)$,

$$d_1=(-1)^{n-r}a_{j\,i_1}\cdot\ldots\cdot a_{i_kj}\cdot b\cdot u_1\cdot\ldots\cdot u_\rho\cdot a_{i_si_{s+m}}\cdot\ldots\cdot a_{i_{s+1}i_s}\cdot v_1\cdot\ldots\cdot v_p,$$

$$d_2=(-1)^{n-r+1}a_{j\,i_1}\cdot\ldots\cdot a_{i_ki_s}\cdot a_{i_si_{s+1}}\dots a_{i_{s+m}j}\cdot b\cdot u_1\cdot\ldots\cdot u_\rho\cdot v_1\cdot\ldots\cdot v_p,$$

$$d_3=(-1)^{n-r+1}a_{ji_1}\cdot\ldots\cdot a_{i_ki_s}\cdot a_{i_s\,i_{s+m}}\cdot\ldots\cdot a_{i_{s+1}j}\cdot b\cdot u_1\cdot\ldots\cdot u_\rho\cdot v_1\cdot\ldots\cdot v_p.$$

Denote $a_{j\,i_1}\cdot\ldots\cdot a_{i_kj}=:x$ and $a_{i_si_{s+1}}\dots a_{i_{s+m}i_s}=:y$, then $\overline{y}=a_{i_si_{s+m}}\dots a_{i_{s+1}i_s}$. Taking into account $a_{i_kj}=a_{i_ki_s}$, $a_{i_{s+m}j}=a_{i_{s+m}i_s}$, $a_{i_{s+1}j}=a_{i_{s+1}i_s}$, we have

$$d+d_1+d_2+d_3=$$

$$=(-1)^{n-r}(x\cdot b\cdot u_1\cdot\ldots\cdot u_\rho\cdot y+x\cdot b\cdot u_1\cdot\ldots\cdot u_\rho\cdot\overline{y}-x\cdot y\cdot b\cdot u_1\cdot\ldots\cdot u_\rho-$$

$$-x\cdot\overline{y}\cdot b\cdot u_1\cdot\ldots\cdot u_\rho)\cdot v_1\cdot\ldots\cdot v_p=(-1)^{n-r}(x\cdot b\cdot u_1\cdot\ldots\cdot u_\rho\cdot(y+\overline{y})-$$

$$-x\cdot(y+\overline{y})\cdot b\cdot u_1\cdot\ldots\cdot u_\rho)\cdot v_1\cdot\ldots\cdot v_p=(-1)^{n-r}(x\cdot b\cdot u_1\cdot\ldots\cdot u_\rho\cdot t(y)-$$

$$-x\cdot t(y)\cdot b\cdot u_1\cdot\ldots\cdot u_\rho)\cdot v_1\cdot\ldots\cdot v_p=0.$$

Thus among the monomials of $\mathrm{rdet}_j\,\mathbf{A}_{.j}\left(\mathbf{a}_{.i}\cdot b\right)$ we find three monomials for d such that the sum of these monomials and d is equal to zero.

If in (11) $m=0$ or $m=1$, we obtain such monomials accordingly:

$$\tilde{d}=(-1)^{n-r}a_{j\,i_1}\cdot\ldots\cdot a_{i_kj}\cdot b\cdot u_1\cdot\ldots\cdot u_\rho\cdot a_{i_si_s}\cdot v_1\cdot\ldots\cdot v_p,$$

$$\breve{d}=(-1)^{n-r}a_{j\,i_1}\cdot\ldots\cdot a_{i_kj}\cdot b\cdot u_1\cdot\ldots\cdot u_\rho\cdot a_{i_si_{s+1}}\cdot a_{i_{s+1}i_s}\cdot v_1\cdot\ldots\cdot v_p.$$

There are the following monomials for them:

$$\tilde{d}_1=(-1)^{n-r+1}a_{j\,i_1}\cdot\ldots\cdot a_{i_ki_s}a_{i_sj}\cdot b\cdot u_1\cdot\ldots\cdot u_\rho\cdot v_1\cdot\ldots\cdot v_p,$$

$$\breve{d}_1 = (-1)^{n-r+1} a_{j\,i_1} \cdot \ldots \cdot a_{i_k i_s} \cdot a_{i_s i_{s+1}} \cdot a_{i_{s+1} j} \cdot b \cdot u_1 \cdot \ldots \cdot u_\rho \cdot v_1 \cdot \ldots \cdot v_p,$$

Taking into account, that $a_{i_k j} = a_{i_k i_s}$, $a_{i_s j} = a_{i_s i_s}$, $a_{i_{s+1} j} = a_{i_{s+1} i_s}$ and $a_{i_s i_s} \in \mathbb{F}$, $a_{i_s i_{s+1}} a_{i_{s+1} i_s} = n\left(a_{i_s i_{s+1}}\right) \in \mathbb{F}$, we obtain

$$\begin{aligned} \tilde{d} + \tilde{d}_1 = (-1)^{n-r} (a_{j\,i_1} \cdot \ldots \cdot a_{i_k j} \cdot b \cdot u_1 \cdot \ldots \cdot u_\rho \cdot a_{i_s i_s} - \\ - a_{j\,i_1} \cdot \ldots \cdot a_{i_k i_s} \cdot a_{i_s j} \cdot b \cdot u_1 \cdot \ldots \cdot u_\rho) \cdot v_1 \cdot \ldots \cdot v_p = 0, \end{aligned}$$

$$\begin{aligned} \breve{d} + \breve{d}_1 = (-1)^{n-r} (a_{j\,i_1} \cdot \ldots \cdot a_{i_k j} \cdot b \cdot u_1 \cdot \ldots \cdot u_\rho \cdot n\left(a_{i_s i_{s+1}}\right) - \\ - a_{j\,i_1} \cdot \ldots \cdot a_{i_k i_s} \cdot n\left(a_{i_s i_{s+1}}\right) \cdot b \cdot u_1 \cdot \ldots \cdot u_\rho) \cdot v_1 \cdot \ldots \cdot v_p = 0. \end{aligned}$$

Hence, the sums of corresponding two monomials of $\mathrm{rdet}_j\, \mathbf{A}_{.j}\,(\mathbf{a}_{.i} \cdot b)$ are equal to zero in this case.

ii) Now suppose that the index i_s is placed in another disjoint cycle than the index j and does not open this cycle,

$$d = (-1)^{n-r} a_{j\,i_1} \ldots a_{i_k j} b\, u_1 \ldots u_\rho a_{i_q i_{q+1}} \ldots a_{i_{s-1} i_s} a_{i_s i_{s+1}} \ldots a_{i_{q-1} i_q} v_1 \ldots v_p,$$

Here we denote by u_τ and v_t products of coefficients whose indices form some disjoint cycles for all $\tau = \overline{1, \rho}$ and $t = \overline{1, p}$ such that $\rho + p = r - 2$ or there are no such products. Now for d there are the following three monomials of $\mathrm{rdet}_j\, \mathbf{A}_{.j}\,(\mathbf{a}_{.i} \cdot b)$,

$$\overset{\frown}{d}_1 = (-1)^{n-r} a_{j\,i_1} \ldots a_{i_k j} b u_1 \ldots u_\rho a_{i_q i_{q-1}} \ldots a_{i_{s+1} i_s} a_{i_s i_{s-1}} \ldots a_{i_{q+1} i_q} v_1 \ldots v_p\ ,$$

$$\overset{\frown}{d}_2 = (-1)^{n-r} a_{j\,i_1} \ldots a_{i_k i_s} a_{i_s i_{s-1}} \ldots \cdot a_{i_{q+1} i_q} a_{i_q i_{q-1}} \ldots a_{i_{s+1} j}\, b u_1 \ldots u_\rho v_1 \ldots v_p\ ,$$

$$\overset{\frown}{d}_3 = (-1)^{n-r} a_{j\,i_1} \ldots a_{i_k i_s} a_{i_s i_{s-1}} \ldots a_{i_{q+1} i_q} a_{i_q i_{q-1}} \ldots a_{i_{s+1} j}\, b\, u_1 \ldots u_\rho v_1 \ldots v_p\ .$$

Denote $a_{j\,i_1} \cdot \ldots \cdot a_{i_k j} = :x$, $a_{i_q i_{q+1}} \cdot \ldots \cdot a_{i_{s-1} i_s} = :\phi$, $a_{i_s i_{s+1}} \cdot \ldots \cdot a_{i_{q-1} i_q} = :\varphi$, then we have $a_{i_s i_{s-1}} \cdot \ldots \cdot a_{i_{q+1} i_q} = \overline{\phi}\ \ a_{i_q i_{q-1}} \cdot \ldots \cdot a_{i_{s+1} i_s} = \overline{\varphi}$. Taking into account that $a_{i_k j} = a_{i_k i_s}$, $a_{i_{s-1} j} = a_{i_{s-1} i_s}$, $a_{i_{s+1} j} = a_{i_{s+1} i_s}$, we obtain

$$\begin{gathered} d + \overset{\frown}{d}_1 + \overset{\frown}{d}_2 + \overset{\frown}{d}_3 = \\ = (-1)^{n-r} (x b u_1 \ldots u_\rho \phi\, \varphi + x b u_1 \ldots u_\rho \overline{\varphi}\, \overline{\phi} - \\ - x \varphi\, \phi b u_1 \ldots u_\rho - x \overline{\phi}\, \overline{\varphi}\, b u_1 \ldots u_\rho) v_1 \ldots v_p = \\ = (-1)^{n-r} (x b u_1 \ldots u_\rho (\phi\, \varphi + \overline{\phi \varphi}) - x (\varphi\, \phi + \overline{\varphi \phi}) b u_1 \ldots u_\rho) v_1 \ldots v_p = \\ = (-1)^{n-r} (x b u_1 \ldots u_\rho t(\phi\, \varphi) - x\, t(\varphi\, \phi) b u_1 \ldots u_\rho) v_1 \ldots v_p. \end{gathered}$$

by the rearrangement property of the trace $\mathrm{t}(\phi \cdot \varphi) = \mathrm{t}(\varphi \cdot \phi) \in F$, we obtain $d + \overset{\frown}{d}_1 + \overset{\frown}{d}_2 + \overset{\frown}{d}_3 = 0$.

(iii) If the indices i_s and j are placed in the same cycle, then we have the following monomials: $\overset{\frown}{d}_1$ or $\breve{d}_1$, d_2, $\tilde{d}_2$, and d_3, $\tilde{d}_3$ as well. As shown above, for each of them there are another one or three monomials of $\mathrm{rdet}_j\, \mathbf{A}_{j\,.}(\mathbf{a}_{i\,.})$ such that the sums of these two or four corresponding monomials are equal to zero.

We have considered all possible kinds of disposition of an entry of the i_sth row as a factor of some monomial d of $\mathrm{rdet}_j\, \mathbf{A}_{j\,.}(\mathbf{a}_{i\,.})$. In each case there exist one or three corresponding monomials for d such that the sum of the two or four monomials is equal to zero respectively. Hence, $\mathrm{rdet}_j\, \mathbf{A}_{.j}\,(\mathbf{a}_{.i} \cdot b) = 0$. ■

Corollary 5.2. *If the matrix* $\mathbf{A}_{.j}\left(\mathbf{a}_{.i}\right)$ *is obtained from a Hermitian matrix* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$ *by replacing of its* j*th column with the* i*th column, then* $\operatorname{rdet}_j \mathbf{A}_{.j}\left(\mathbf{a}_{.i}\right) = 0$ *for all* $i, j = \overline{1, n}$ *such that* $i \neq j$.

The proof follows immediately from Theorem 5.5, by putting $b = 1$.

Theorem 5.6. *If the matrix* $\mathbf{A}_{i.}\left(b \cdot \mathbf{a}_{j.}\right)$ *is obtained from a Hermitian matrix* $\mathbf{A} \in M\left(n, \mathbb{H}\right)$ *by replacing of its* i*th row with the* j*th row left-multiplied by* $b \in \mathbb{H}$*, then* $\operatorname{cdet}_i \mathbf{A}_{i.}\left(b \cdot \mathbf{a}_{j.}\right) = 0$ *for all* $i, j = \overline{1, n}$ *such that* $i \neq j$.

The proof is similar to the proof of Theorem 5.5.

Corollary 5.3. *If the matrix* $\mathbf{A}_{i.}\left(\mathbf{a}_{j.}\right)$ *is obtained from a Hermitian matrix* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$ *by replacing of its* i*th row with the* j*th row, then* $\operatorname{cdet}_i \mathbf{A}_{i.}\left(\mathbf{a}_{j.}\right) = 0$ *for all* $i, j = \overline{1, n}$ *such that* $i \neq j$.

The proof follows immediately from Theorem 5.6, by putting $b = 1$.

Lemma 5.1. *If the matrix* $\mathbf{A}_{.i}\left(\mathbf{a}_{.i} \cdot b\right)$ *is obtained from a Hermitian matrix* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$ *by right-multiplying of its* i*th column by* $b \in \mathbb{H}$*, then for all* $i = \overline{1, n}$ *we have*

$$\operatorname{rdet}_i \mathbf{A}_{.i}\left(b \cdot \mathbf{a}_{.i}\right) = \operatorname{rdet}_i \mathbf{A}_{.i}\left(\mathbf{a}_{.i} \cdot b\right) = \det \mathbf{A} \cdot b$$

.

Proof. Consider some monomial d of $\mathbf{A}_{.i}\left(\mathbf{a}_{.i} \cdot b\right)$ for $i = \overline{1,n}$, where the matrix $\mathbf{A}_{.i}\left(\mathbf{a}_{.i} \cdot b\right)$ is obtained from a Hermitian matrix $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$ by right-multiplying of its ith column by $b \in \mathbb{H}$ Denote $i_{k_1} := i$.

$$\begin{gathered} d = (-1)^{n-r} a_{i_{k_1} i_{k_1+1}} \dots a_{i_{k_1+l_1} i_{k_1}} b \, a_{i_{k_2} i_{k_2+1}} \dots a_{i_{k_2+l_2} i_{k_2}} \dots \times \\ \times a_{i_{k_r} i_{k_r+1}} \dots a_{i_{k_r+l_r} i_{k_r}} = (-1)^{n-r} h_1 \cdot h_2 \cdot \ldots \cdot h_r, \end{gathered}$$

where $h_s = a_{i_{k_s} i_{k_s+1}} \cdot \ldots \cdot a_{i_{k_s+l_s} i_{k_s}}$ for all $\left(s = \overline{1, r} \right)$. If $l_s = 1$, then $h_s = a_{i_{k_s} i_{k_s+1}} \cdot a_{i_{k_s+1} i_{k_s}} = n(a_{i_{k_s} i_{k_s+1}}) \in \mathbb{F}$, and if $l_s = 0$, then $h_s = a_{i_{k_s} i_{k_s}} \in F$. Suppose there is s such that $l_s \geq 2$. The index permutation σ of d forms a direct products of disjoint cycles and its cycle notation is left-ordered. Denote by $\sigma_s\left(i_{k_s}\right) := \left(i_{k_s} i_{k_s+1} \dots i_{k_s+l_s}\right)$ a cycle which corresponds to a factor h_s. Then $\sigma_s^{-1}\left(i_{k_s}\right) := \left(i_{k_s} i_{k_s+l_s} i_{k_s+1} \dots i_{k_s+1}\right)$ is the cycle which is inverse to $\sigma_s\left(i_{k_s}\right)$ and corresponds to the factor $\overline{h_s}$. There are 2^{p-1} monomials of $\mathbf{A}_{.i}\left(\mathbf{a}_{.i} \cdot b\right)$ such that their indices permutations form the direct products of the disjoint cycles $\sigma_s\left(i_{k_s}\right)$ or $\sigma_s^{-1}\left(i_{k_s}\right)$ for all $\left(s = \overline{1, r} \right)$ and keeping their ordering from 1 to r. We have $p = r - \rho$, where ρ is the number of the cycles of the first and second orders. Then by lemma 4.1 for the sum C_1 of these monomials and d we obtain,

$$C = (-1)^{n-r} b \cdot \alpha t(h_{\nu_1}) \, \dots \, t(h_{\nu_p}),$$

where $\alpha \in F$ is a product of the factors whose indices form the cycles of the first and second orders. Since $\mathrm{t}(h_{\nu_k}) \in \mathbb{F}$ for all $\nu_k \in \{1, \dots, r\}$ and $k = \overline{1, p}$, then b commutes with $\mathrm{t}(h_{\nu_k}) \in \mathbb{F}$ for all $\nu_k \in \{1, \dots, r\}$ and $k = \overline{1, p}$. Then we obtain $\operatorname{rdet}_i \mathbf{A}_{.i}\left(\mathbf{a}_{.i} \cdot b\right) = \operatorname{rdet}_i \mathbf{A} \cdot b = b \cdot \det \mathbf{A}$.

By theorem 3.2 we have $\operatorname{rdet}_i \mathbf{A}_{.i}\left(b \cdot \mathbf{a}_{.i}\right) = b \cdot \operatorname{rdet}_i \mathbf{A} = b \cdot \det \mathbf{A}$ as well. ■

Lemma 5.2. *If* $\mathbf{A}_{i\,.}\left(b\cdot\mathbf{a}_{i\,.}\right)$ *is obtained from a Hermitian matrix* $\mathbf{A}\in\mathrm{M}\left(n,\mathbb{H}\right)$ *by left-multiplying of its* i*th row by* $b\in\mathbb{H}$*, then for all* $i=\overline{1,n}$ *we have*

$$\mathrm{cdet}_i\mathbf{A}_{i\,.}\left(b\cdot\mathbf{a}_{i\,.}\right)=\mathrm{cdet}_i\mathbf{A}_{i\,.}\left(\mathbf{a}_{i\,.}\cdot b\right)=b\cdot\det\mathbf{A}$$

The proof is similar to the proof of Lemma 5.1

From Theorems 5.1, 5.6 and basic properties of the row and column determinants for arbitrary matrices we have the following theorem.

Theorem 5.7. *If the* i*th row of a Hermitian matrix* $\mathbf{A}\in\mathrm{M}\left(n,\mathbb{H}\right)$ *is replaced with a left linear combination of its other rows, i.e.* $\mathbf{a}_{i.}=c_1\mathbf{a}_{i_1.}+\ldots+c_k\mathbf{a}_{i_k.}$*, where* $c_l\in\mathbb{H}$ *for all* $l=\overline{1,k}$ *and* $\{i,i_l\}\subset I_n$*, then*

$$\mathrm{rdet}_i\,\mathbf{A}_{i\,.}\left(c_1\mathbf{a}_{i_1.}+\ldots+c_k\mathbf{a}_{i_k.}\right)=\mathrm{cdet}_i\,\mathbf{A}_{i\,.}\left(c_1\mathbf{a}_{i_1.}+\ldots+c_k\mathbf{a}_{i_k.}\right)=0.$$

Theorem 5.8. *If the* j*th column of a Hermitian matrix* $\mathbf{A}\in\mathrm{M}\left(n,\mathbb{H}\right)$ *is replaced with a right linear combination of its other columns, i.e.* $\mathbf{a}_{.j}=\mathbf{a}_{.j_1}c_1+\ldots+\mathbf{a}_{.j_k}c_k$*, where* $c_l\in\mathbb{H}$ *for all* $l=\overline{1,k}$ *and* $\{j,j_l\}\subset J_n$*, then*

$$\mathrm{cdet}_j\,\mathbf{A}_{.j}\left(\mathbf{a}_{.j_1}c_1+\ldots+\mathbf{a}_{.j_k}c_k\right)=\mathrm{rdet}_j\,\mathbf{A}_{.j}\left(\mathbf{a}_{.j_1}c_1+\ldots+\mathbf{a}_{.j_k}c_k\right)=0.$$

The proof follows immediately from Theorems 5.2, 5.5 and basic properties of the row and column determinants for arbitrary matrices as well.

From Theorems 5.7, 5.8 and basic properties of the row and column determinants for arbitrary matrices we obtain the following theorems respectively.

Theorem 5.9. *If the* i*th row of a Hermitian matrix* $\mathbf{A}\in\mathrm{M}\left(n,\mathbb{H}\right)$ *is added a left linear combination of its other rows, then*

$$\begin{gathered}\mathrm{rdet}_i\,\mathbf{A}_{i\cdot}\left(\mathbf{a}_{i.}+c_1\cdot\mathbf{a}_{i_1.}+\ldots+c_k\cdot\mathbf{a}_{i_k.}\right)=\\=\mathrm{cdet}_i\,\mathbf{A}_{i\cdot}\left(\mathbf{a}_{i.}+c_1\cdot\mathbf{a}_{i_1.}+\ldots+c_k\cdot\mathbf{a}_{i_k.}\right)=\det\mathbf{A},\end{gathered}$$

where $c_l\in\mathbb{H}$ *for all* $l=\overline{1,k}$ *and* $\{i,i_l\}\subset I_n$*.*

Theorem 5.10. *If the* j*th column of a Hermitian matrix* $\mathbf{A}\in\mathrm{M}\left(n,\mathbb{H}\right)$ *is added a right linear combination of its other columns, then*

$$\begin{gathered}\mathrm{cdet}_j\,\mathbf{A}_{.j}\left(\mathbf{a}_{.j}+\mathbf{a}_{.j_1}c_1+\ldots+\mathbf{a}_{.j_k}c_k\right)=\\=\mathrm{rdet}_j\,\mathbf{A}_{.j}\left(\mathbf{a}_{.j}+\mathbf{a}_{.j_1}c_1+\ldots+\mathbf{a}_{.j_k}c_k\right)=\det\mathbf{A},\end{gathered}$$

where $c_l\in\mathbb{H}$ *for all* $l=\overline{1,k}$ *and* $\{j,j_l\}\subset J_n$*.*

6. Diagonalization of Hermitian Matrices

Suppose the matrix $\mathbf{E}_{ij}=(e_{pq})_{n\times n}$ such that $e_{pq}=\left\{\begin{array}{l}1,\,p=i,\,q=j,\\0,\,p\neq i,\,q\neq j.\end{array}\right.$

Definition 6.1. *The matrix* $\mathbf{P}_{ij}(b) := \mathbf{I}+b\cdot\mathbf{E}_{ij} \in \mathrm{M}(n,\mathbb{H})$ *for* $i \neq j$ *is called an elementary unimodular matrix, where* $\mathbf{I}$ *is the identity matrix. Matrices* $\mathbf{P}_{ij}(b)$ *for* $i \neq j$ *and for all* $b \in \mathbb{H}$ *generate the unimodular group* $\mathrm{SL}(n,\mathbb{H})$*, its elements is called the unimodular matrices.*

Theorem 6.1. *If* $\mathbf{A} \in \mathrm{M}(n,\mathbb{H})$ *is a Hermitian matrix and* $\mathbf{P}_{ij}\left(b\right)$ *is an elementary unimodular matrix, then* $\det \mathbf{A} = \det\left(\mathbf{P}_{ij}\left(b\right)\cdot\mathbf{A}\cdot\mathbf{P}^{*}_{ij}\left(b\right)\right)$.

Proof. First note that for all $\mathbf{U} \in \mathrm{M}(n,\mathbb{H})$ and a Hermitian matrix $\mathbf{A}$, the matrix $\mathbf{U}^{*}\mathbf{A}\mathbf{U}$ is Hermitian as well. Really, $(\mathbf{U}^{*}\mathbf{A}\mathbf{U})^{*} = \mathbf{U}^{*}\mathbf{A}^{*}\mathbf{U} = \mathbf{U}^{*}\mathbf{A}\mathbf{U}$. Multiplying a matrix $\mathbf{A}$ by $\mathbf{P}_{ij}\left(b\right)$ on the left adds the jth row left-multiplied by b to the ith row. Whereas multiplying a matrix $\mathbf{A}$ by $\mathbf{P}^{*}_{ij}\left(b\right)$ on the right adds the jth column right-multiplied by $\overline{b}$ to the jth column. Therefore,

$$\mathbf{P}_{ij}\left(b\right)\cdot\mathbf{A}\cdot\mathbf{P}^{*}_{ij}\left(b\right) =$$

$$\begin{pmatrix} a_{11} & \dots & a_{1i}+a_{1j}\overline{b} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1}+ba_{j1} & \dots & (ba_{jj}+a_{ij})\overline{b}+ba_{ji}+a_{ii} & \dots & a_{in}+ba_{jn} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{ni}+a_{nj}\,\overline{b} & \dots & a_{nn} \end{pmatrix}$$

Then by Theorems 3.4 and 3.5, we have

$$\det\left(\mathbf{P}_{ij}\left(b\right)\cdot\mathbf{A}\cdot\mathbf{P}^{*}_{ij}\left(b\right)\right) = \mathrm{cdet}_{i}\left(\mathbf{P}_{ij}\left(b\right)\cdot\mathbf{A}\cdot\mathbf{P}^{*}_{ij}\left(b\right)\right) =$$

$$= \mathrm{cdet}_{i}\begin{pmatrix} a_{11} & ... & a_{1i} & ... & a_{1n} \\ ... & ... & ... & ... & ... \\ a_{i1}+ba_{j1} & ... & a_{ii}+ba_{ji} & ... & a_{in}+ba_{jn} \\ ... & ... & ... & ... & ... \\ a_{n1} & ... & a_{ni} & ... & a_{nn} \end{pmatrix} +$$

$$+\mathrm{cdet}_{i}\begin{pmatrix} a_{11} & ... & a_{1j}\overline{b} & ... & a_{1n} \\ ... & ... & ... & ... & ... \\ a_{i1}+ba_{j1} & ... & (ba_{jj}+a_{ij})\overline{b} & ... & a_{in}+ba_{jn} \\ ... & ... & ... & ... & ... \\ a_{n1} & ... & a_{nj}\,\overline{b} & ... & a_{nn} \end{pmatrix} =$$

$$= \mathrm{cdet}_{i}\mathbf{A} + \mathrm{cdet}_{i}\mathbf{A}_{i.}(b\cdot\mathbf{a}_{j.}) + \mathrm{cdet}_{i}\mathbf{A}_{.\,i}(\mathbf{a}_{.j})\cdot\overline{b}+$$

$$+\mathrm{cdet}_{i}\begin{pmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} \\ ... & ... & ... & ... & ... \\ ba_{j1} & ... & ba_{jj} & ... & ba_{jn} \\ ... & ... & ... & ... & ... \\ a_{n1} & ... & a_{nj} & ... & a_{nn} \end{pmatrix}\cdot\overline{b}.$$

The matrix $\begin{pmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} \\ ... & ... & ... & ... & ... \\ ba_{j1} & ... & ba_{jj} & ... & ba_{jn} \\ ... & ... & ... & ... & ... \\ a_{n1} & ... & a_{nj} & ... & a_{nn} \end{pmatrix} = (\mathbf{A}_{.\,i}(\mathbf{a}_{.j}))_{i\,.}(b\mathbf{a}_{j\,.})$ is obtained from $\mathbf{A}$ by replacing its ith column with the jth column, and then by replacing the ith row of the obtained matrix with its jth row left-multiplied by b. The ith row of $\mathbf{A}_{.\,i}(\mathbf{a}_{.j}))_{i\,.}(b\mathbf{a}_{j\,.})$ is $b\mathbf{a}_{j\,.}$ and its jth row is $\mathbf{a}_{j.}$. Then by Theorem 5.6, we get $\text{cdet}_i(\mathbf{A}_{.\,i}(\mathbf{a}_{.j}))_{i\,.}(b\mathbf{a}_{j\,.}) = 0$. Furthermore by Theorem 5.2 we have $\text{cdet}_i\mathbf{A}_{.\,i}(\mathbf{a}_{.j}) = 0$, and by Theorem 5.6 we obtain $\text{cdet}_i\mathbf{A}_{i\,.}(b \cdot \mathbf{a}_{j.}) = 0$.

Finally, we have $\det\left(\mathbf{P}_{ij}\left(b\right) \cdot \mathbf{A} \cdot \mathbf{P}^*_{ij}\left(b\right)\right) = \text{cdet}_i\mathbf{A} = \det\mathbf{A}$. ■

Theorem 6.2. *If* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$ *is a Hermitian matrix and* $\mathbf{U} \in \mathrm{SL}(n, \mathbf{H})$*, then*

$$\det\mathbf{A} = \det\left(\mathbf{U} \cdot \mathbf{A} \cdot \mathbf{U}^*\right).$$

Proof. We claim that there exist $\{\mathbf{P}_1, \ldots, \mathbf{P}_k\} \subset \mathrm{SL}(n, \mathbb{H})$ and $k \in \mathbb{N}$ for $\mathbf{U} \in \mathrm{SL}\left(n, \mathbb{H}\right)$ such that $\mathbf{U} = \mathbf{P}_k \cdot \ldots \cdot \mathbf{P}_1$. Then $\mathbf{U}^* = \mathbf{P}^*_1 \cdot \ldots \cdot \mathbf{P}^*_k$.

We prove the theorem by induction on k.

i) The case $k = 1$ has been proved Theorem 6.1.

ii) Suppose the theorem is valid for $k - 1$. That is $\mathbf{U} = \mathbf{P}_{k-1} \cdot \ldots \cdot \mathbf{P}_1$ and

$$\det\mathbf{A} = \det\left(\mathbf{P}_{k-1} \cdot \ldots \cdot \mathbf{P}_1 \cdot \mathbf{A} \cdot \mathbf{P}^*_1 \cdot \ldots \cdot \mathbf{P}^*_{k-1}\right).$$

Denote $\tilde{\mathbf{A}} := \mathbf{P}_{k-1} \cdot \ldots \cdot \mathbf{P}_1 \cdot \mathbf{A} \cdot \mathbf{P}^*_1 \cdot \ldots \cdot \mathbf{P}^*_{k-1}$. As shown in Theorem 6.1 a matrix $\tilde{\mathbf{A}}$ is Hermitian.

iii) If now $\mathbf{U} = \mathbf{P}_k \cdot \mathbf{P}_{k-1} \ldots \cdot \mathbf{P}_1$, then

$$\det\left(\mathbf{U} \cdot \mathbf{A} \cdot \mathbf{U}^*\right) = \det\left(\mathbf{P}_k \cdot \tilde{\mathbf{A}} \cdot \mathbf{P}^*_k\right) = \det\tilde{\mathbf{A}} = \det\mathbf{A}.■$$

Lemma 6.1. *If* $\mathrm{U} \in \mathrm{SL}(n, \mathbb{H})$*, then* $\left\{\mathbf{U}^{-1}, \mathbf{U}^*\right\} \in \mathrm{SL}(n, \mathbb{H})$.

Proof. Let $\mathbf{U}$ is a unimodular matrix and $\mathbf{U} = \prod\limits_{k=1}^{m} \mathbf{P}_k$, where $\mathbf{P}_k = \mathbf{P}_{ij}\left(b_k\right)$ are unimodular matrices, (i.e. $\exists m \in \mathbf{N}, \forall k = \overline{1, m}, \exists b_k \in \mathbb{H}, \exists i \in I_n, \exists j \in I_n, i \neq j$). Then

$\mathbf{P}^{-1}_k = \mathbf{P}^{-1}_{ij}\left(b_k\right) = \mathbf{P}_{ij}(-b_k) \in \mathrm{SL}(n, \mathbb{H})$, $\prod_{k=m}^{1} \mathbf{P}^{-1}_k = \mathbf{U}^{-1} \in \mathrm{SL}(n, \mathbb{H})$,

$\mathbf{P}^*_k = \mathbf{P}^*_{ij}\left(b_k\right) = \mathbf{P}_{ji}\left(\overline{b_k}\right) \in \mathrm{SL}(n, \mathbb{H})$, $\prod_{k=m}^{1} \mathbf{P}^*_k = \mathbf{U}^* \in \mathrm{SL}(n, \mathbb{H})$. ■

Theorem 6.3. *If* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$ *is a Hermitian matrix, then there exist* $\mathbf{U} \in \mathrm{SL}(n, \mathbb{H})$ *and* $\mu_i \in \mathbb{F}$ *for all* $i = \overline{1, n}$*, such that* $\mathbf{U} \cdot \mathbf{A} \cdot \mathbf{U}^* = \mathrm{diag}(\mu_1, \ldots, \mu_n)$*, where* $\mathrm{diag}(\mu_1, \ldots, \mu_n)$ *is a diagonal matrix. Then* $\det\mathbf{A} = \mu_1 \cdot \ldots \cdot \mu_n$.

Proof. Consider the first column of a Hermitian matrix $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$. It is possible the following cases.

i) If $a_{11} \neq 0$, then $\mu_1 = a_{11} \in \mathbb{F}$. By sequentially left-multiplying the matrix $\mathbf{A}$ by elementary unimodular matrices $\mathbf{P}_{i1}\left(-\frac{a_{i1}}{\mu_1}\right)$ for all $i = \overline{2, n}$, we obtain zero for all entries of the first column save for diagonal. Since $\overline{-\frac{a_{i1}}{\mu_1}} = -\frac{a_{1i}}{\mu_1}$, then $\mathbf{P}^*_{i1}\left(-\frac{a_{i1}}{\mu_1}\right) = \mathbf{P}_{1i}\left(-\frac{a_{1i}}{\mu_1}\right)$. By sequentially right-multiplying the matrix $\mathbf{A}$ by elementary unimodular matrices $\mathbf{P}^*_{i1}\left(-\frac{a_{i1}}{\mu_1}\right)$, we get zero for all entries of the first row save for diagonal. Due to Theorem 6.1 the obtained matrix is Hermitian as well.

ii) Suppose $a_{11} = 0$ and there exists $i \in I_n$ such that $a_{i1} \neq 0$. Having multiplied the matrix $\mathbf{A}$ by elementary unimodular matrices $\mathbf{P}_{1i}\left(a_{1i}\right)$ on the left and by $\mathbf{P}_{i1}\left(a_{i1}\right)$ on the right, we get the matrix $\tilde{\mathbf{A}}$ with an entry $\tilde{a}_{11} = \mathrm{n}(a_{i1})\left(2 + a_{i\,i}\right) \in \mathbb{F}$. Let now $\mu_1 = \tilde{a}_{11}$. Again by sequentially multiplying the matrix $\tilde{\mathbf{A}}$ by $\mathbf{P}_{i1}\left(-\frac{\tilde{a}_{i1}}{\mu_1}\right)$ on the left and by $\mathbf{P}^*_{i1}\left(-\frac{\tilde{a}_{i1}}{\mu_1}\right)$ for all $i = \overline{2, n}$, on the right, we obtain the matrix with zero for all entries of the first row and column save for diagonal.

iii) If $i \in I_n$ for all $a_{i1} = 0$, then put $\mu_1 = a_{11}$.

Having carried through the described procedure for all diagonal entries and entries of corresponding rows and columns by means of a finite number of multiplications the Hermitian matrix $\mathbf{A}$ by elementary unimodular matrices $\mathbf{P}_k = \mathbf{P}_{ij}\left(b_k\right)$ on the left and by $\mathbf{P}^*_k = \mathbf{P}_{ji}\left(\overline{b_k}\right)$ on the right, we obtain the diagonal matrix with diagonal entries $\mu_i \in \mathbb{F}$ for all $i = \overline{1, n}$. Suppose $\mathbf{U} = \prod\limits_k \mathbf{P}_k$, then by Theorem 6.2 we finally obtain

$$\det(\mathbf{U} \cdot \mathbf{A} \cdot \mathbf{U}^*) = \det\left(\mathrm{diag}\left(\mu_1, \ldots, \mu_\mathrm{n}\right)\right) = \mu_1 \cdot \ldots \cdot \mu_\mathrm{n}. \blacksquare$$

Corollary 6.1. *If* $\mathbf{A}, \mathbf{B}$ *are Hermitian over* $\mathbb{H}$ *and* $\mathbf{AB} = \mathbf{BA}$, *then* $\det\left(\mathbf{AB}\right) = \det \mathbf{A} \cdot \det \mathbf{B}$.

Proof. We have $\mathbf{A} = \mathbf{A}^*$ and $\mathbf{B} = \mathbf{B}^*$. Hence, $\left(\mathbf{AB}\right)^* = \mathbf{B}^*\mathbf{A}^* = \mathbf{BA} = \mathbf{AB}$. By theorem 6.3 there exist $\mathbf{U}, \mathbf{V} \subset \mathrm{SL}(n, \mathbb{H})$ and $\mu_i, \eta_i \subset \mathbb{F}$ for all $i = \overline{1, n}$, such that $\mathbf{U} \cdot \mathbf{A} \cdot \mathbf{U}^* = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ and $\mathbf{V} \cdot \mathbf{A} \cdot \mathbf{V}^* = \mathrm{diag}(\eta_1, \ldots, \eta_n)$. Then we obtain

$$\begin{gathered}\det\left(\mathbf{AB}\right) = \det\left(\mathbf{U} \cdot \mathbf{A} \cdot \mathbf{U}^*\mathbf{V} \cdot \mathbf{B} \cdot \mathbf{V}^*\right) = \\ \det\left(diag\left(\mu_1, \ldots, \mu_n\right) diag\left(\eta_1, \ldots, \eta_n\right)\right) = \\ = \mu_1 \cdot \eta_1 \cdot \ldots \cdot \mu_n \cdot \eta_n = \mu_1 \cdot \ldots \cdot \mu_n \cdot \eta_1 \cdot \ldots \cdot \eta_n = \det \mathbf{A} \cdot \det \mathbf{B}.\end{gathered}$$

■

7. The Inverse of a Hermitian Matrix

Definition 7.1. *A Hermitian matrix* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$ *is called nonsingular, if* $\det \mathbf{A} \neq 0$.

Theorem 7.1. *There exist a unique right inverse matrix* $\left(R\mathbf{A}\right)^{-1}$ *and a unique left inverse matrix* $\left(L\mathbf{A}\right)^{-1}$ *of a nonsingular Hermitian matrix* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$, *where* $\left(R\mathbf{A}\right)^{-1} = \left(L\mathbf{A}\right)^{-1} =: \mathbf{A}^{-1}$ *and*

$$\left(R\mathbf{A}\right)^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\ R_{12} & R_{22} & \cdots & R_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{pmatrix}, \tag{12}$$

$$\left(L\mathbf{A}\right)^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\ L_{12} & L_{22} & \cdots & L_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ L_{1n} & L_{2n} & \cdots & L_{nn} \end{pmatrix}, \tag{13}$$

where R_{ij}, L_{ij} *are right and left* ij*-th cofactor of* $\mathbf{A}$ *respectively for all* $i, j = \overline{1, n}$.

Proof. Let $\mathbf{B} = \mathbf{A} \cdot (R\mathbf{A})^{-1}$. We obtain the entries of $\mathbf{B}$ by multiplying matrices for all $i = \overline{1, n}$

$$b_{ii} = (\det \mathbf{A})^{-1} \sum_{j=1}^{n} a_{ij} \cdot R_{ij} = (\det \mathbf{A})^{-1} \operatorname{rdet}_i \mathbf{A} = \frac{\det \mathbf{A}}{\det \mathbf{A}} = 1,$$

and for all $i \neq j$

$$b_{ij} = (\det \mathbf{A})^{-1} \sum_{s=1}^{n} a_{is} \cdot R_{js} = (\det \mathbf{A})^{-1} \operatorname{rdet}_j \mathbf{A}_{j.}\left(\mathbf{a}_{i.}\right).$$

If $i \neq j$, then by Theorem 5.1 $\operatorname{rdet}_j \mathbf{A}_{j.}\left(\mathbf{a}_{i.}\right) = 0$. Consequently $b_{ij} = 0$. Thus $\mathbf{B} = \mathbf{I}$ and $(R\mathbf{A})^{-1}$ is the right inverse of the Hermitian matrix $\mathbf{A}$.

Suppose $\mathbf{D} = (L\mathbf{A})^{-1}\, \mathbf{A}$. We again get the entries of $\mathbf{D}$ by multiplying matrices, for all $i = \overline{1, n}$:

$$d_{ii} = (\det \mathbf{A})^{-1} \sum_{i=1}^{n} L_{ij} \cdot a_{ij} = (\det \mathbf{A})^{-1} \operatorname{cdet}_j \mathbf{A} = \frac{\det \mathbf{A}}{\det \mathbf{A}} = 1,$$

and for all $i \neq j$

$$d_{ij} = (\det \mathbf{A})^{-1} \sum_{s=1}^{n} L_{s\,i} \cdot a_{sj} = (\det \mathbf{A})^{-1} \operatorname{cdet}_i \mathbf{A}_{i.}\left(\mathbf{a}_{j.}\right).$$

If $i \neq j$, then by Theorem 5.2 $\operatorname{cdet}_i \mathbf{A}_{i.}\left(\mathbf{a}_{j.}\right) = 0$. Therefore $d_{ij} = 0$ for all $i \neq j$. Thus $\mathbf{D} = \mathbf{I}$ and $(L\mathbf{A})^{-1}$ is the left inverse of the Hermitian matrix $\mathbf{A}$.

The equality $(R\mathbf{A})^{-1} = (L\mathbf{A})^{-1}$ is immediate from the well-known fact that if there exists an inverse matrix over an arbitrary skew field, then it is unique.■

Theorem 7.2. *If* $\mathbf{A}$ *is a nonsingular Hermitian matrix* $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$, *then* $\det \mathbf{A}^{-1} = (\det \mathbf{A})^{-1}$.

Proof. Whereas $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$ is a nonsingular Hermitian matrix, then by theorem 6.3 there exist $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{F}$ and an unimodular matrix $\mathbf{U} \in \mathrm{SL}(n, \mathbb{H})$ such that $\mathbf{A} = \mathbf{U} \cdot \operatorname{diag}\left(\lambda_1, \ldots, \lambda_n\right) \cdot \mathbf{U}^*$ and $\det \mathbf{A} = \lambda_1 \cdot \ldots \cdot \lambda_n$. The matrix $\mathbf{A}^{-1}$ is Hermitian as well. Then we obtain

$$\mathbf{A}^{-1} = \left(\mathbf{U} \cdot \operatorname{diag}\left(\lambda_1, \ldots, \lambda_n\right) \cdot \mathbf{U}^*\right)^{-1} = \left(\mathbf{U}^*\right)^{-1} \cdot \operatorname{diag}\left(\lambda_1^{-1}, \ldots, \lambda_n^{-1}\right) \cdot \mathbf{U}^{-1}$$

By lemma 6.1, we have $\left\{\mathbf{U}^{-1}, \left(\mathbf{U}^*\right)^{-1}\right\} \subset \mathrm{SL}(n, \mathbb{H})$. Then by theorem 7.1, we obtain

$$\begin{aligned}&\det \mathbf{A}^{-1} = \\ &\det\left(\operatorname{diag}\left(\lambda_1^{-1}, \ldots, \lambda_n^{-1}\right)\right) = \lambda_1^{-1} \cdot \ldots \cdot \lambda_n^{-1} = \left(\lambda_1 \cdot \ldots \cdot \lambda_n\right)^{-1} = (\det \mathbf{A})^{-1}.\end{aligned}$$

■

Remark 7.1. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is a nonsingular Hermitian matrix, then its classic adjoint matrix may be represented as* $\mathrm{Adj}\,\mathbf{A} = (L_{ij})_{n\times n}$ *or* $\mathrm{Adj}\,\mathbf{A} = (R_{ij})_{n\times n}$. *We have for a nonsingular Hermitian matrix* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*:*

$$\mathbf{A}^{-1} = \frac{\mathrm{Adj}\ [\mathbf{A}]}{\det \mathbf{A}}.$$

Since $\mathbf{A}^{-1}$ *is Hermitian, then* $\mathbf{A}$, $\mathbf{A}^{-1}$, *and* $\mathbf{A}$, $Adj\,[\mathbf{A}]$ *are commutative pairs of Hermitian matrices. Then by Corollary 6.1, we obtain*

$$\det \mathbf{A} \cdot \left(\mathbf{A}^{-1}\mathbf{A}\right) = \mathrm{Adj}\,[\mathbf{A}] \cdot \mathbf{A} = \mathrm{diag}\,(\det \mathbf{A}, \ldots, \det \mathbf{A})\,.$$

From here we have $\det\left(\mathrm{Adj}\,[\mathbf{A}]\right) = (\det \mathbf{A})^{n-1}$.

The following criterion of invertibility of Hermitian matrix completes this subsection.

Theorem 7.3. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is a nonsingular Hermitian matrix, then the following propositions are equivalent*

i) $\mathbf{A}$ *is invertibility, i.e.* $\mathbf{A} \in GL\,(n, S)\,;$

ii) $\det \mathbf{A} \neq 0$;

iii) the rows of $\mathbf{A}$ *are left-linearly independent;*

iiii) the columns of $\mathbf{A}$ *are right-linearly independent.*

Proof. The equivalence of the propositions i) and ii) follows from Theorems 7.1 and 7.2. The equivalence of the propositions ii) and iii) follows from Theorem 5.7. The equivalence of the propositions ii) and iiii) follows from Theorem 5.8 as well. ■

Remark 7.2. *The determinant of a Hermitian matrix satisfies Axioms 1 and 3 from Definition 1.1. It follows from Theorem 7.3, Theorems 3.4 and 3.5 and Corollary 5.1 respectively.*

8. Properties of the Corresponding Hermitian Matrices

Denote by $\mathbb{H}^{m\times n}$ a set of $m \times n$ matrices with entries in $\mathbb{H}$.

Definition 8.1. *For an arbitrary matrix* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *the matrix* $\mathbf{A}^*\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is called its left corresponding Hermitian and* $\mathbf{A}\mathbf{A}^* \in \mathrm{M}(m, \mathbb{H})$ *is called its right corresponding Hermitian matrix.*

Theorem 8.1. *If the* j*th column of* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *is right-multiplied by* $b \in \mathbb{H}$ *for all* $j = \overline{1, n}$, *then the determinant of its left corresponding Hermitian matrix is multiplied by* $\mathrm{n}(b)$.

Proof. The matrix $\mathbf{A}_{.j}\left(\mathbf{a}_{.j} \cdot b\right)$ is obtained from $\mathbf{A} \in \mathbb{H}^{m\times n}$ by right-multiplying of its jth column on $b \in \mathbb{H}$ for all $j = \overline{1, n}$. Then we have

$$\left(\mathbf{A}_{.j}\left(\mathbf{a}_{.j} \cdot b\right)\right)^* = \mathbf{A}^*_{j.}\left(\overline{b} \cdot \mathbf{a}_{j.}\right),$$

where $\mathbf{A}^*_{j.}\left(\overline{b}\cdot\mathbf{a}_{j.}\right)$ is obtained from $\mathbf{A}\in S^{n\times m}$ by left-multiplying of its jth row on $\overline{b}$. Then we obtain

$$\mathbf{A}^*_{j.}\left(\overline{b}\cdot\mathbf{a}_{j.}\right)\cdot\mathbf{A}_{.j}\left(\mathbf{a}_{.j}\cdot b\right)=$$

$$=\begin{array}{cc}\left(\begin{array}{ccccc}
\sum\limits_{k=1}^{m}\overline{a_{k1}}a_{k1} & \ldots & \sum\limits_{k=1}^{m}\overline{a_{k1}}a_{kj}\cdot b & \ldots & \sum\limits_{k=1}^{m}\overline{a_{k1}}a_{kn}\\
\ldots & \ldots & \ldots & \ldots & \ldots\\
\sum\limits_{k=1}^{m}\overline{b}\cdot\overline{a_{kj}}a_{k1} & \ldots & \sum\limits_{k=1}^{m}\overline{b}\cdot\overline{a_{kj}}a_{kj}\cdot b & \ldots & \sum\limits_{k=1}^{m}\overline{b}\cdot\overline{a_{kj}}a_{kn}\\
\ldots & \ldots & \ldots & \ldots & \ldots\\
\sum\limits_{k=1}^{m}\overline{a_{kn}}a_{k1} & \ldots & \sum\limits_{k=1}^{m}\overline{a_{kn}}a_{kj}\cdot b & \ldots & \sum\limits_{k=1}^{m}\overline{a_{kn}}a_{kn}
\end{array}\right) & j-th\\
j-th &
\end{array}$$

The matrix $\mathbf{A}^*_{j.}\left(\overline{b}\cdot\mathbf{a}_{j.}\right)\cdot\mathbf{A}_{.j}\left(\mathbf{a}_{.j}\cdot b\right)$ is Hermitian. Then by Theorem 3.2 and Lemma 5.1, we have

$$\det\left(\mathbf{A}^*_{j.}\left(\overline{b}\cdot\mathbf{a}_{j.}\right)\cdot\mathbf{A}_{.j}\left(\mathbf{a}_{.j}\cdot b\right)\right)=\overline{b}\cdot\mathrm{rdet}_j\left(\mathbf{A}^*\cdot\mathbf{A}_{.j}\left(\mathbf{a}_{.j}\cdot b\right)\right)=$$
$$=\overline{b}\cdot\mathrm{rdet}_j\left(\mathbf{A}^*\mathbf{A}\right)\cdot b=\overline{b}\cdot\det\left(\mathbf{A}^*\mathbf{A}\right)\cdot b=\mathrm{n}\left(b\right)\det\left(\mathbf{A}^*\mathbf{A}\right).$$

■

Theorem 8.2. *If ith row of $\mathbf{A}\in\mathbb{H}^{m\times n}$ is left-multiplied by $b\in\mathbb{H}$ for all $j=\overline{1,n}$, then the determinant of its right corresponding Hermitian matrix is multiplied by* $\mathrm{n}(b)$.

The proof is similar to the proof of Theorem 8.2

Theorem 8.3. *If the matrix $\mathbf{A}\in\mathbb{H}^{m\times n}$ has two identical columns, then* $\det(\mathbf{A}^*\mathbf{A})=0$.

Proof. Let the matrix $\mathbf{A}\in\mathbb{H}^{m\times n}$ has two identical columns, sth and tth, i.e. $a_{is}=a_{it}$ for all $i\in I_m$ such that $s\neq t$ and $\{s,t\}\subset J_n$. Then the Hermitian adjoint matrix $\mathbf{A}^*$ has two identical rows, sth and tth. Consider the matrix $\mathbf{A}^*\mathbf{A}$.

$$\mathbf{A}^*\mathbf{A}=$$

$$\begin{pmatrix}
\overline{a_{11}} & \cdots & \overline{a_{s1}} & \cdots & \overline{a_{t1}} & \cdots & \overline{a_{m1}}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\overline{a_{1s}} & \cdots & \overline{a_{ss}} & \cdots & \overline{a_{ts}} & \cdots & \overline{a_{ms}}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\overline{a_{1s}} & \cdots & \overline{a_{st}} & \cdots & \overline{a_{tt}} & \cdots & \overline{a_{mt}}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\overline{a_{1n}} & \cdots & \overline{a_{sn}} & \cdots & \overline{a_{ts}} & \cdots & \overline{a_{mn}}
\end{pmatrix}\times\begin{pmatrix}
a_{11} & \cdots & a_{1s} & \cdots & a_{1t} & \cdots & a_{1n}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
a_{s1} & \cdots & a_{ss} & \cdots & a_{st} & \cdots & a_{sn}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
a_{t1} & \cdots & a_{ts} & \cdots & a_{tt} & \cdots & a_{tn}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
a_{m1} & \cdots & a_{ms} & \cdots & a_{mt} & \cdots & a_{mn}
\end{pmatrix}=$$

$$= \begin{pmatrix} \sum\limits_{k=1}^{m} \overline{a_{k1}} \cdot a_{k1} & \cdots & \sum\limits_{k=1}^{m} \overline{a_{k1}} \cdot a_{k\,s} & \cdots & \sum\limits_{k=1}^{m} \overline{a_{k1}} \cdot a_{k\,t} & \cdots & \sum\limits_{k=1}^{m} \overline{a_{k1}} \cdot a_{k\,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum\limits_{k=1}^{m} \overline{a_{k\,s}} \cdot a_{k1} & \cdots & \sum\limits_{k=1}^{m} \overline{a_{k\,s}} \cdot a_{k\,s} & \cdots & \sum\limits_{k=1}^{m} \overline{a_{k\,s}} \cdot a_{kt} & \cdots & \sum\limits_{k=1}^{m} \overline{a_{ks}} \cdot a_{kn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum\limits_{k=1}^{m} \overline{a_{k\,t}} \cdot a_{k1} & \cdots & \sum\limits_{k=1}^{m} \overline{a_{kt}} \cdot a_{k\,s} & \cdots & \sum\limits_{k=1}^{m} \overline{a_{kt}} \cdot a_{kt} & \cdots & \sum\limits_{k=1}^{m} \overline{a_{kt}} \cdot a_{k\,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum\limits_{k=1}^{m} \overline{a_{k\,m}} \cdot a_{k1} & \cdots & \sum\limits_{k=1}^{m} \overline{a_{k\,m}} \cdot a_{k\,s} & \cdots & \sum\limits_{k=1}^{m} \overline{a_{k\,m}} \cdot a_{k\,t} & \cdots & \sum\limits_{k=1}^{m} \overline{a_{k\,m}} \cdot a_{k\,n} \end{pmatrix}$$

Since for all $k \in I_m$ we have $a_{k\,s} = a_{k\,t}$, then $\sum\limits_{k=1}^{m} \overline{a_{k\,l}} \cdot a_{k\,s} = \sum\limits_{k=1}^{m} \overline{a_{k\,l}} \cdot a_{k\,t}$ for all $l = \overline{1, n}$. Therefore the Hermitian matrix $\mathbf{A}^*\mathbf{A}$ has two identical rows, sth and tth as well. Then by Corollary 5.1, det $\mathbf{A}^*\mathbf{A} = 0$. ■

Theorem 8.4. *If the matrix* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *has two identical rows, then* $\det(\mathbf{A}^*\mathbf{A}) = 0$.

The proof is similar to the proof of Theorem 8.3

Theorem 8.5. *If* i*th column of* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *is replaced with its* j*th column right-multiplied by an arbitrary* $b \in \mathbb{H}$ *and* $i \neq j$*, then* $det\mathbf{A}^*\mathbf{A} = 0$.

Proof. Let $\mathbf{A}_{.\,i}\,(\mathbf{a}_{.\,j} \cdot b)$ is a matrix obtained from $\mathbf{A} \in \mathbb{H}^{m\times n}$ by replaced its ith column with its jth column right-multiplied by an arbitrary $b \in \mathbb{H}$ and $i \neq j$ for all $j, i = \overline{1, n}$. Then its Hermitian adjoint matrix is a matrix $\mathbf{A}^*_{i\,.}\,\left(\overline{b} \cdot \mathbf{a}_{j.}\right)$. This matrix is obtained from $\mathbf{A} \in \mathbb{H}^{m\times n}$ by replaced its ith row with its jth row left-multiplied by $\overline{b}$. Then by Theorem 8.1, we obtain

$$\det\left(\mathbf{A}^*_{i\,.}\,\left(\overline{b} \cdot \mathbf{a}_{j.}\right) \cdot \mathbf{A}_{.\,i}\,(\mathbf{a}_{.\,j} \cdot b)\right) = n\,(b) \cdot \det\left(\mathbf{A}^*_{i\,.}\,(\mathbf{a}_{j.}) \cdot \mathbf{A}_{.\,i}\,(\mathbf{a}_{.\,j})\right).$$

Therefore the matrix $\mathbf{A}_{.\,i}\,(\mathbf{a}_{.\,j})$ has two identical columns, i.e. $a_{k\,i} = a_{k\,j}$ for all $k = \overline{1, m}$, then by Theorem 8.3 we obtain

$$\det\left(\mathbf{A}^*_{i\,.}\,(\mathbf{a}_{j.}) \cdot \mathbf{A}_{.\,i}\,(\mathbf{a}_{.\,j})\right) = 0.$$

■

Theorem 8.6. *If* i*th row of* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *is replaced with its* j*th row left-multiplied by an arbitrary* $b \in \mathbb{H}$ *and* $i \neq j$*, then* $det\mathbf{A}\mathbf{A}^* = 0$.

The proof is similar to the proof of Theorem 8.5 and follows immediately from Theorems 8.2 and 8.4.

Theorem 8.7. *If an arbitrary column of* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *is a right linear combination of its other columns, then* det $\mathbf{A}^*\mathbf{A} = 0$.

Proof. Let the jth column of $\mathbf{A}$ is a right linear combination of other columns. That is there exist $b_1, \ldots, b_k$, such that $b_l \in \mathbb{H}$ for all $l = \overline{1,k}$ and

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1i_1} \cdot b_1 + \ldots + a_{1j_k} \cdot b_k & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mj_1} \cdot b_1 + \ldots + a_{mj_k} \cdot b_k & \cdots & a_{mn} \end{pmatrix}, \\ \qquad\qquad j-th$$

where $j_l \in \{1, \ldots, j-1, j+1, \ldots, n\}$. Then jth row of $\mathbf{A}^*$ is the left linear combination of the rows $j_1, \ldots, j_k$ with coefficients $\overline{b_1}, \ldots, \overline{b_k}$. That is,

$$\mathbf{A}^* = \begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{m1}} \\ \cdots & \cdots & \cdots \\ \overline{b_1} \cdot \overline{a_{1j_1}} + \ldots + \overline{b_k} \cdot \overline{a_{1j_k}} & a_{32} & \overline{b_1} \cdot \overline{a_{mj_1}} + \ldots + \overline{b_k} \cdot \overline{a_{mj_k}} \\ \cdots & \cdots & \cdots \\ \overline{a_{1n}} & \cdots & \overline{a_{mn}} \end{pmatrix} \quad j-th$$

Then the jth column of the left corresponding Hermitian matrix $\mathbf{A}^*\mathbf{A}$ is

$$\begin{pmatrix} \sum\limits_{s=1}^{m} \left(\overline{b_1}\overline{a_{sj_1}} + \cdots + \overline{b_k \cdot}\overline{a_{sj_k}} \right) a_{s1} \\ \cdots \\ \sum\limits_{s=1}^{m} \left(\overline{b_1} \cdot \overline{a_{sj_1}} + \ldots + \overline{b_k \cdot}\overline{a_{sj_k}} \right) \left(a_{sj_1} b_1 + \ldots + a_{sj_k} b_k \right) \\ \cdots \\ \sum\limits_{s=1}^{m} \overline{a_{sn}} \left(a_{sj_1} b_1 + \cdots + a_{sj_k} b_k \right) \end{pmatrix}.$$

The jth row of $\mathbf{A}^*\mathbf{A}$ is

$$\begin{pmatrix} \sum\limits_{s=1}^{m} \overline{a_{s1}} \cdot \left(a_{sj_1} \cdot b_1 + \ldots + a_{sj_k} \cdot b_k \right) \\ \cdots \\ \sum\limits_{s=1}^{m} \left(\overline{b_1} \cdot \overline{a_{sj_1}} + \ldots + \overline{b_k \cdot}\overline{a_{sj_k}} \right) \left(a_{sj_1} b_1 + \ldots + a_{sj_k} b_k \right) \\ \cdots \\ \sum\limits_{s=1}^{m} \left(\overline{b_1} \cdot \overline{a_{sj_1}} + \cdots + \overline{b_k}\overline{a_{sj_k}} \right) a_{sn} \end{pmatrix}^T$$

The j-th column of $\mathbf{A}^*\mathbf{A}$ is the right linear combination of the column $j_1, \ldots, j_k$ Then by Corollary 5.1, we obtain

$$\det \mathbf{A}^*\mathbf{A} = 0.$$

Theorem 8.8. *If an arbitrary row of* $\mathbf{A} \in \mathbb{H}^{m \times n}$ *is a left linear combination of its other rows, then* $\det \mathbf{A}\mathbf{A}^* = 0$.

9. The Criterion of a Singularity of the Corresponding Hermitian Matrix

Definition 9.1. *Row vectors*

$$\begin{aligned} &\mathbf{a}_{1.} = (a_{11}, \ldots, a_{1n}), \\ &\vdots \\ &\mathbf{a}_{m.} = (a_{m1}, \ldots, a_{mn}), \end{aligned} \tag{14}$$

are said to be left linearly independent, if there are $\{b_1, \ldots, b_m\} \subset \mathbf{H}$ *(which are not all zero) such that*

$$b_1 \cdot \mathbf{a}_{1.} + \ldots + b_m \cdot \mathbf{a}_{m.} = \mathbf{0}. \tag{15}$$

where $a_{i\,j} \in \mathbf{H}$ *for all* $i \in I_m$ *and* $j \in J_n$*, and* $\mathbf{0}$ *is the zero row vector.*

Definition 9.2. *The row vectors (14) is called the left linear dependent, if the equality (15) is possible only when all* $b_1, \ldots, b_m$ *are zero.*

Definition 9.3. *Column vectors*

$$\mathbf{a}_{.1} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \quad \ldots, \quad \mathbf{a}_{.n} = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}, \tag{16}$$

are said to be right linearly independent, if there are $\{c_1, \ldots, c_m\} \subset \mathbf{H}$ *(which are not all zero) such that*

$$\mathbf{a}_{.j_1} \cdot c_1 + \ldots + \mathbf{a}_{.j_n} \cdot c_n = \mathbf{0}. \tag{17}$$

where $a_{i\,j} \in \mathbf{H}$ *for all* $i \in I_m$ *and* $j \in J_n$*, and* $\mathbf{0}$ *is the zero column vector.*

Definition 9.4. *The column vectors (16) is called the right linear dependent, if the equality (17) is possible only when all* $c_1, \ldots, c_m$ *are zero.*

We have immediately the following linear independence criterions which are similar to the commutative case.

Theorem 9.1. *Row vectors is left linear dependent* **iff** *one of them can be written as a left linear combination of the others.*

Theorem 9.2. *Column vectors is right linear dependent* **iff** *one of them can be written as a right linear combination of the others.*

Since the principal submatrices of a Hermitian matrix are also Hermitian, then the basis principal minor may be defined in this noncommutative case as well.

Definition 9.5. *Let Hermitian matrix* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *has a nonzero principal minor of order* $r \leq n$ *and all principal minors of order more than* r *(if there exist) are equal zeros. Then the natural number* r *is called the rank by principal minors of* $\mathbf{A}$*. A principal nonzero minor of order* r *is said to be basic, rows and columns which form this minor are called basic as well.*

Definition 9.6. *If the rows and the columns with indices $i_1, \ldots, i_r$ of the Hermitian matrix $\mathbf{A}^*\mathbf{A}$ are basis, then the rows with indices $i_1, \ldots, i_r$ of $\mathbf{A}^*$ are called basis and the columns with indices $i_1, \ldots, i_r$ of $\mathbf{A} \in \mathbb{H}^{m\times n}$ are called the basis ones as well.*

Theorem 9.3. *The basis rows of $\mathbf{A}^*\mathbf{A}$ and $\mathbf{A}^* \in \mathbb{H}^{n\times m}$ are left-linearly independent, and the basis columns of $\mathbf{A}^*\mathbf{A}$ and $\mathbf{A} \in \mathbb{H}^{m\times n}$ are right-linearly independent.*

Proof. Suppose that basis rows of $\mathbf{A}^*\mathbf{A}$ are left-linearly dependent. Then by Theorem 9.1 one of them can be written as a left linear combination of the others. Subtracting the linear combination from this row, we obtain a row that consists of zeros only. Then by Theorem 3.1 the basis principal minor of $\mathbf{A}^*\mathbf{A}$ is equal to zero, but this contradicts its definition.

Suppose that basis columns of $\mathbf{A}^*\mathbf{A}$ are right-linearly dependent. Then by Theorem 9.2 one of them can be written as a right linear combination of the others. Subtracting the linear combination from this column, we obtain a column that consists of zeros only. Then by Theorem 3.1 the basis principal minor of $\mathbf{A}^*\mathbf{A}$ is equal to zero, but this contradicts its definition.

Suppose that basis rows of $\mathbf{A}^*$ are left-linearly dependent. Then by Theorem 9.1 one of them can be written as a left linear combination of the others. Hence by Theorem 8.8 the basis principal minor of $\mathbf{A}^*\mathbf{A}$ is equal to zero, but this contradicts its definition as well.

Suppose finally that basis columns of $\mathbf{A}^*$ are right-linearly dependent. Then by Theorem 9.2 one of them can be written as a right linear combination of the others. Hence by Theorem 8.7 the basis principal minor of $\mathbf{A}^*\mathbf{A}$ is equal to zero, but this contradicts its definition as well. ■

Theorem 9.4. *An arbitrary column of $\mathbf{A} \in \mathbb{H}^{m\times n}$ is a right linear combination of its basis columns.*

Proof. If columns with indices $i_1, \ldots, i_r$ are the basis columns of $\mathbf{A}$, then the basis principal minor of $\mathbf{A}^*\mathbf{A} =: \left(d_{ij}\right)_{n\times n}$ is placed on the crossing of its columns and rows with indices $i_1, \ldots, i_r$ as well. Denote by $\mathbf{M}$ the matrix of the basis principal minor. Supplement it by the $(r+1)$th row and column consisting of corresponding entries of the j-th row and column of $\mathbf{A}^*\mathbf{A}$ respectively. Suppose $j \in \{i_1, \ldots, i_r\}$. The obtained matrix is denoted by $\mathbf{D}_j$.

$$\mathbf{D}_j = \begin{pmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_r} & d_{i_1 j} \\ \cdots & \cdots & \cdots & \cdots \\ d_{i_r i_1} & \cdots & d_{i_r i_r} & d_{i_r j} \\ d_{j\, i_1} & \cdots & d_{j\, i_r} & d_{j\, j} \end{pmatrix}$$

Since the Hermitian matrix $\mathbf{D}_j$ contains two coinciding columns, by Corollary 5.1 we obtain that

$$\det \mathbf{D}_j = \operatorname{cdet}_j \mathbf{D}_j = \sum_{l=1}^{r} L_{i_l\, j} \cdot d_{i_l\, j} + L_{j\, j} \cdot d_{j\, j} = 0,$$

where $L_{i_l j}$ is the left $i_l j$th cofactor of $\mathbf{D}_j$. Whereas $L_{j\,j} = \det \mathbf{M} \neq 0$, we get

$$d_{j\,j} = -\sum_{l=1}^{r} \left(\det \mathbf{M}\right)^{-1} L_{i_l j} \cdot d_{i_l j} \quad \text{for all} \quad j \in \{i_1, \ldots, i_r\}. \tag{18}$$

Now suppose that $j \notin \{i_1, \dots, i_k, i_{k+1}, \dots, i_r\}$ and $i_k < j < i_{k+1}$. Consider the matrix $\mathbf{D}_j$ obtained from $\mathbf{M}$ by supplementing it by the jth row and column:

$$\mathbf{D}_j = \begin{pmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_k} & d_{i_1 j} & d_{i_1 i_{k+1}} & \cdots & d_{i_1 i_r} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ d_{i_k i_1} & \cdots & d_{i_k i_k} & d_{i_k j} & d_{i_k i_{k+1}} & \cdots & d_{i_k i_r} \\ d_{j\, i_1} & \cdots & d_{j\, i_k} & d_{j\, j} & d_{j\, i_{k+1}} & \cdots & d_{j\, i_r} \\ d_{i_{k+1} i_1} & \cdots & d_{i_{k+1} i_k} & d_{i_{k+1} j} & d_{i_{k+1} i_{k+1}} & \cdots & d_{i_{k+1} i_r} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ d_{i_r i_1} & \cdots & d_{i_r i_k} & d_{i_r j} & d_{i_r i_{k+1}} & \cdots & d_{i_r i_r} \end{pmatrix}$$

The matrix $\mathbf{D}_j$ is Hermitian in this case as well. Then we have

$$\det \mathbf{D}_j = \operatorname{cdet}_j \mathbf{D}_j = \sum_{l=1}^{r} L_{i_l\, j} \cdot d_{i_l\, j} + L_{j\, j} \cdot d_{j\, j} = 0.$$

Since $L_{j\, j} = \det \mathbf{M} \neq 0$, then

$$d_{jj} = -\sum_{l=1}^{r} \left(\det \mathbf{M}\right)^{-1} L_{i_l j} \cdot d_{i_l j}, \quad j \notin \{i_1, \dots, i_r\} \subset I_n. \tag{19}$$

Combining (18) and (19), we obtain $d_{j\, j} = -\sum\limits_{l=1}^{r} \left(\det \mathbf{M}\right)^{-1} L_{i_l j} \cdot d_{i_l j}$ for all $j = \overline{1, n}$. If $-\left(\det \mathbf{M}\right)^{-1} L_{i_l j} := \mu_l$, then $d_{j\, j} = \sum\limits_{l=1}^{r} \mu_l \cdot d_{i_l j}$. Since $d_{j\, j} = \sum\limits_{k=1}^{m} \overline{a_{k\, j}} a_{k\, j}$ and $d_{i_l j} = \sum\limits_{k=1}^{m} \overline{a_{k\, i_l}} a_{k\, j}$, then

$$\sum_{k=1}^{m} \overline{a_{k\, j}} a_{k\, j} = \sum_{l=1}^{r} \mu_l \sum_{k=1}^{m} \overline{a_{k\, i_l}} a_{k\, j} = \sum_{k=1}^{m} \sum_{l=1}^{r} \mu_l \overline{a_{k\, i_l}} a_{k\, j}.$$

Hence, $\overline{a_{k\, j}} = \sum\limits_{l=1}^{r} \mu_l \overline{a_{k\, i_l}}$ and so $a_{k\, j} = \sum\limits_{l=1}^{r} a_{k\, i_l} \overline{\mu_l}$ $(\forall k = \overline{1, m})$. Therefore, an arbitrary column of the matrix $\mathbf{A}$ is the right linear combination of its basis columns with the coefficients $\overline{\mu_1}, \dots, \overline{\mu_r}$, i.e.:

$$\mathbf{a}_{.\, i_1} \cdot \overline{\mu_1} + \ldots + \mathbf{a}_{.\, i_r} \cdot \overline{\mu_r} = \mathbf{a}_{.\, j} \text{ for all } i_l \in I_n, \text{ for all } l = \overline{1, r}.$$

■

The following theorem is proved in a similar manner.

Theorem 9.5. *An arbitrary row of* $\mathbf{A} \in \mathbb{H}^{m \times n}$ *is a left linear combination of its basis rows.*

Theorem 9.6. *The right linearly dependence of columns of* $\mathbf{A} \in \mathbb{H}^{m \times n}$ *or the left linearly dependence of rows of* $\mathbf{A}^*$ *is the necessary and sufficient condition for* $\det \mathbf{A}^* \mathbf{A} = 0$.

Proof. (Necessity) If the columns of $\mathbf{A} \in \mathbb{H}^{m\times n}$ are right-linearly dependent, then by Theorem 9.2 one of them can be written as a right linear combination of the others. Hence, by Theorem 8.7 we have $\det \mathbf{A}^*\mathbf{A} = 0$.

Similarly, if the rows of $\mathbf{A}^*$ are left-linearly dependent, then by Theorem 9.1 one of them can be written as a left linear combination of the others. Hence, by Theorem 8.8 we have $\det \mathbf{A}^*\mathbf{A} = 0$ as well.

(Sufficiency) If $\det \mathbf{A}^*\mathbf{A} = 0$, then by Theorem 7.3 its columns are right-linearly dependent. Hence, an its basis principal minor has the order $r < n$. Then at least one of the columns of $\mathbf{A} \in \mathbb{H}^{m\times n}$ is not basic and at least one of the rows of $\mathbf{A}^*$ is not basic as well. By Theorem 9.4 this column is a right linear combination of the other column of $\mathbf{A}$ and by Theorem 9.5 this row is a right linear combination of the other rows of $\mathbf{A}^*$. So by Theorem 9.2 the columns of $\mathbf{A} \in \mathbb{H}^{m\times n}$ are right-linearly dependent. By Theorem 9.1 the rows of $\mathbf{A}^*$ are left-linearly dependent as well. ■

Definition 9.7. *If r is the maximum number of right-linearly independent columns of $\mathbf{A} \in \mathbb{H}^{m\times n}$, then r is called the column rank of the matrix $\mathbf{A}$.*

Definition 9.8. *If r is the maximum number of left-linearly independent columns of $\mathbf{A} \in \mathbb{H}^{m\times n}$, then r is called the row rank of the matrix $\mathbf{A}$? denote by* $\mathrm{rank}\mathbf{A}$.

It is well-known that the column rank of an arbitrary matrix over skew field is equal to its row rank. Whereas we can define the rank of a matrix over the quaternion algebra with division, as the maximum number of left-linearly independent rows or right-linearly independent columns.

Theorem 9.7. *A rank by principal minors of $\mathbf{A}^*\mathbf{A}$ is equal to its rank and a rank of of $\mathbf{A} \in \mathbb{H}^{m\times n}$.*

Proof. Let a rank by principal minors of $\mathbf{A}^*\mathbf{A}$ is r, then by Theorem 9.3 r basic n-dimension column of $\mathbf{A}$ are right-linearly independent. Let for certainty they are $\mathbf{a}_{.1}, ..., \mathbf{a}_{.n}$. Consider $\mathrm{span}\,(\mathbf{a}_{.1}, ..., \mathbf{a}_{.n}) \subset \mathbb{H}^n$, where $\mathbb{H}^n$ is the right vector space. Since by Theorem 9.4 an arbitrary column of $\mathbf{A}$ is right-linearly combination of its basic columns, then the basic columns are a basis of $\mathrm{span}\,(\mathbf{a}_{.1}, ..., \mathbf{a}_{.n})$. Then any $r+1$ vectors of $\mathrm{span}\,(\mathbf{a}_{.1}, ..., \mathbf{a}_{.n})$ are right-linearly dependent. So any $r+1$ columns of $\mathbf{A}$ are right-linearly dependent as well and r is the maximal number of right-linearly independent columns of $\mathbf{A}$, i.e. $\mathrm{rank}\,\mathbf{A} = r$.

Similarly, by Theorem 9.3 r basic n-dimension column of $\mathbf{A}^*\mathbf{A}$ are right-linearly independent. Denote by $\tilde{\mathbf{a}}_{.k}$ a column of $\mathbf{A}^*\mathbf{A}$ for all $k = 1, ..., n$. Consider $\mathrm{span}\,(\tilde{\mathbf{a}}_{.1}, ..., \tilde{\mathbf{a}}_{.n}) \subset \mathbb{H}^n$. Since by Theorem 9.4 an arbitrary column of $\mathbf{A}$ is right-linearly combination of its basic columns, then as shown in Theorem 8.7 an arbitrary column of $\mathbf{A}^*\mathbf{A}$ is right-linearly combination of its basic columns. So the basic columns are a basis of $\mathrm{span}\,(\tilde{\mathbf{a}}_{.1}, ..., \tilde{\mathbf{a}}_{.n})$ and $\dim\,(\mathrm{span}\,(\tilde{\mathbf{a}}_{.1}, ..., \tilde{\mathbf{a}}_{.n})) = r$. Then any $r+1$ vectors of $\mathrm{span}\,(\tilde{\mathbf{a}}_{.1}, ..., \tilde{\mathbf{a}}_{.n})$ are right-linearly dependent. So any $r+1$ columns of $\mathbf{A}^*\mathbf{A}$ are right-linearly dependent as well and r is the maximal number of right-linearly independent columns $\mathbf{A}^*\mathbf{A}$, i.e. $\mathrm{rank}\,(\mathbf{A}^*\mathbf{A}) = r$. ■

The following theorem is proved in a similar manner.

Theorem 9.8. *A rank by principal minors of $\mathbf{A}\mathbf{A}^*$ is equal to its rank and a rank of of $\mathbf{A} \in \mathbb{H}^{m\times n}$.*

10. Properties of the Double Determinant of a Quaternion Square Matrix

Theorem 10.1. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, then* $\det \mathbf{A}\mathbf{A}^* = \det \mathbf{A}^*\mathbf{A}$.

Proof. Suppose $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$. It is easy to see that

$$\det \begin{pmatrix} -\mathbf{I} & \mathbf{A} \\ \mathbf{A}^* & \mathbf{0} \end{pmatrix} = \det \begin{pmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^* & -\mathbf{I} \end{pmatrix}.$$

The matrix $\begin{pmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$ can be represented as a product of n^2 elementary unimodular $2n \times 2n$ matrices, i.e. for all $k = \overline{1, n^2}$ there exists $i = \overline{1, n}$ and $j = \overline{n+1, n^2}$ and there exists $\mathbf{P}_k = \mathbf{P}_{ij}^{(k)}(a_{ij})$ such that

$$\begin{pmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} = \prod_k \mathbf{P}_k.$$

Thus,

$$\begin{pmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \in \mathrm{SL}(2n, \mathbb{H}).$$

In a similar manner

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}^* & \mathbf{I} \end{pmatrix} \in \mathrm{SL}(2n, \mathbb{H}).$$

From this by Theorem 6.2, we have

$$(-1)^n \det \mathbf{A}\mathbf{A}^* = \det \begin{pmatrix} \mathbf{A}\mathbf{A}^* & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix} =$$
$$= \det \left(\begin{pmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^* & -\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}^* & \mathbf{I} \end{pmatrix} \right) = \det \begin{pmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^* & -\mathbf{I} \end{pmatrix} =$$
$$= \det \begin{pmatrix} -\mathbf{I} & \mathbf{A} \\ \mathbf{A}^* & \mathbf{0} \end{pmatrix} = \det \left(\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}^* & \mathbf{I} \end{pmatrix} \begin{pmatrix} -\mathbf{I} & \mathbf{A} \\ \mathbf{A}^* & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \right) =$$
$$= \det \begin{pmatrix} \mathbf{A}\mathbf{A}^* & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix} = (-1)^n \det \mathbf{A}^*\mathbf{A}. \blacksquare$$

Definition 10.1. *For* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *the determinant of its corresponding Hermitian matrix is called its double determinant, i.e.*

$$\mathrm{ddet}\mathbf{A} := \det(\mathbf{A}^*\mathbf{A}) = \det(\mathbf{A}\mathbf{A}^*).$$

Theorem 10.2. *If* $\forall\{\mathbf{A}, \mathbf{B}\} \subset \mathrm{M}(n, \mathbb{H})$*, then* $\mathrm{ddet}(\mathbf{A} \cdot \mathbf{B}) = \mathrm{ddet}\mathbf{A} \cdot \mathrm{ddet}\mathbf{B}$.

Proof. Due to Theorem 6.3 for the Hermitian matrix $\mathbf{A}^*\mathbf{A}$, there exists $\mathbf{U} \in \mathrm{SL}(n, \mathbb{H})$ such that

$$\mathbf{U}^* \cdot \mathbf{A}^*\mathbf{A} \cdot \mathbf{U} = (\mathbf{A} \cdot \mathbf{U})^* \cdot \mathbf{A} \cdot \mathbf{U} = \mathrm{diag}(\alpha_1, \ldots, \alpha_n),$$

where $\alpha_i \in \mathbf{R}$. If $\mathbf{A} \cdot \mathbf{U} = (q_{ij})_{n \times n}$, then $\alpha_i = \sum\limits_k \overline{q_{ki}} q_{ki} = \sum\limits_k n(q_{ki}) \in \mathbf{R}_+$ for all $i = \overline{1, n}$, where $\mathbf{R}_+$ is the set of the nonnegative real numbers. Therefore for any $\alpha_i \in \mathbf{R}_+$

there exists $\sqrt{\alpha_i} \in \mathbf{R}_+$ for all $i = \overline{1,n}$. By virtue of $(\mathbf{U}^*)^{-1} = (\mathbf{U}^{-1})^*$ for Hermitian $(\mathbf{U}^{-1}\mathbf{B})^*(\mathbf{U}^{-1}\mathbf{B})$ there exist $\mathbf{W} \in \mathrm{SL}\,(n, \mathbb{H})$ and $\beta_i \in \mathbf{R}_+$ for all $i = \overline{1,n}$ such that $\mathbf{W}^*(\mathbf{U}^{-1}\mathbf{B})^*(\mathbf{U}^{-1}\mathbf{B})\mathbf{W} = \mathrm{diag}(\beta_1, \ldots, \beta_n)$. Hence by Theorems 6.3 and 10.1, we obtain

$$\begin{aligned}
&\mathrm{ddet}(\mathbf{A}\cdot\mathbf{B}) = \det(\mathbf{B}^*(\mathbf{A}^*\mathbf{A})\mathbf{B}) = \det(\mathbf{B}^*(\mathbf{U}^*)^{-1}\mathbf{U}^*(\mathbf{A}^*\mathbf{A})\mathbf{U}\mathbf{U}^{-1}\mathbf{B})\\
&= \det\left(\left(\mathbf{U}^{-1}\mathbf{B}\right)^*\mathrm{diag}\,(\alpha_1, \ldots, \alpha_n)\,\mathbf{U}^{-1}\mathbf{B}\right)\\
&= \det\left((\mathrm{diag}\left(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}\right)\mathbf{U}^{-1}\mathbf{B})^*(\mathrm{diag}\left(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}\right)\mathbf{U}^{-1}\mathbf{B})\right)\\
&= \det\left((\mathrm{diag}\left(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}\right)\mathbf{U}^{-1}\mathbf{B})(\mathrm{diag}\left(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}\right)\mathbf{U}^{-1}\mathbf{B})^*\right)\\
&= \det\left(\mathrm{diag}\left(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}\right)(\mathbf{U}^{-1}\mathbf{B})(\mathbf{U}^{-1}\mathbf{B})^*\mathrm{diag}\left(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}\right)\right)\\
&= \det\left(\mathrm{diag}\left(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}\right)(\mathbf{W}^{-1})^*\mathrm{diag}\,(\beta_1, \ldots, \beta_n)\,\mathbf{W}^{-1}\times\right.\\
&\quad \left.\mathrm{diag}\left(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}\right)\right) = \det\left(\left((\mathbf{W}^{-1})^T\right)^*\mathrm{diag}\left(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}\right)\times\right.\\
&\quad \left.\times\mathrm{diag}\,(\beta_1, \ldots, \beta_n)\cdot\mathrm{diag}\left(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}\right)(\mathbf{W}^{-1})^T\right)\\
&= \det\left(\mathrm{diag}\left(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}\right)\cdot\mathrm{diag}\,(\beta_1, \ldots, \beta_n)\,\mathrm{diag}\left(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}\right)\right)\\
&= \alpha_1\cdot\ldots\cdot\alpha_n\cdot\beta_1\cdot\ldots\cdot\beta_n = \det\mathbf{A}\cdot\det\mathbf{B} = \det\mathbf{B}\cdot\det\mathbf{A}.\blacksquare
\end{aligned}$$

Remark 10.1. *The proofs of Theorems 10.1 and 10.2 are similarly to the proofs in [5, p.533], and they differ by using different determinant functionals.*

Remark 10.2. *Unfortunately, if non-Hermitian matrix is not full rank, then nothing can be said about singularity of its row and column determinant. We show it in the following example*

Example 10..1. *Consider the matrix* $\mathbf{A} = \begin{pmatrix} i & j \\ j & -i \end{pmatrix}$. *Its second row is obtained from the first row by left-multiplying by* k. *Then by Theorem 8.6* $\mathrm{ddet}\mathbf{A} = 0$. *Indeed,*

$$\mathbf{A}^*\mathbf{A} = \begin{pmatrix} -i & -j \\ -j & i \end{pmatrix}\cdot\begin{pmatrix} i & j \\ j & -i \end{pmatrix} = \begin{pmatrix} 2 & -2k \\ 2k & 2 \end{pmatrix},$$

then $\mathrm{ddet}\mathbf{A} = 4 + 4k^2 = 0$. *But*

$$\mathrm{cdet}_1\mathbf{A} = \mathrm{cdet}_2\mathbf{A} = \mathrm{rdet}_1\mathbf{A} = \mathrm{rdet}_2\mathbf{A} = -i^2 - j^2 = 2.$$

At the same time $\mathrm{rank}\,\mathbf{A} = 1$, *that corresponds to Theorem 9.7.*

11. Determinantal Representation of the Inverse Matrix

Definition 11.1. *Suppose that* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *and*

$$\mathrm{ddet}\mathbf{A} = \mathrm{cdet}_j\left(\mathbf{A}^*\mathbf{A}\right) = \sum_i \mathbb{L}_{ij}\cdot a_{ij},$$

for all $j = \overline{1,n}$. *Then* $\mathbb{L}_{ij}$ *is called the left double* ij*-th cofactor of* $\mathbf{A}$.

Definition 11.2. *Suppose that* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *and*

$$\mathrm{ddet}\mathbf{A} = \mathrm{rdet}_i\left(\mathbf{A}\mathbf{A}^*\right) = \sum_j a_{ij}\cdot\mathbb{R}_{ij},$$

for all $i = \overline{1,n}$. *Then* $\mathbb{R}_{ij}$ *is called the right double* ij*-th cofactor of* $\mathbf{A}$.

Theorem 11.1. *The necessary and sufficient condition of invertibility of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is* $\mathrm{ddet}\mathbf{A} \neq 0$*. Then there exists* $\mathbf{A}^{-1} = (L\mathbf{A})^{-1} = (R\mathbf{A})^{-1}$*, where*

$$(L\mathbf{A})^{-1} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^* = \frac{1}{\mathrm{ddet}\mathbf{A}} \begin{pmatrix} \mathbb{L}_{11} & \mathbb{L}_{21} & \ldots & \mathbb{L}_{n1} \\ \mathbb{L}_{12} & \mathbb{L}_{22} & \ldots & \mathbb{L}_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{L}_{1n} & \mathbb{L}_{2n} & \ldots & \mathbb{L}_{nn} \end{pmatrix} \tag{20}$$

$$(R\mathbf{A})^{-1} = \mathbf{A}^*(\mathbf{A}\mathbf{A}^*)^{-1} = \frac{1}{\mathrm{ddet}\mathbf{A}^*} \begin{pmatrix} \mathbb{R}_{11} & \mathbb{R}_{21} & \ldots & \mathbb{R}_{n1} \\ \mathbb{R}_{12} & \mathbb{R}_{22} & \ldots & \mathbb{R}_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{R}_{1n} & \mathbb{R}_{2n} & \ldots & \mathbb{R}_{nn} \end{pmatrix} \tag{21}$$

and

$$\mathbb{L}_{ij} = \mathrm{cdet}_j(\mathbf{A}^*\mathbf{A})_{.j}\left(\mathbf{a}^*_{.i}\right), \ \ \mathbb{R}_{ij} = \mathrm{rdet}_i(\mathbf{A}\mathbf{A}^*)_{i.}\left(\mathbf{a}^*_{j.}\right),$$

for all $i, j = \overline{1, n}$.

Proof. Necessity. Suppose that there exists the inverse matrix $\mathbf{A}^{-1}$ of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$. By virtue of $\mathrm{rank}\,\mathbf{A} \geq \mathrm{rank}(\mathbf{A}^{-1}\mathbf{A}) = \mathrm{rank}\,\mathbf{I} = n$, we have $\mathrm{rank}\,\mathbf{A} = n$. Thus, the columns of $\mathbf{A}$ are right linearly independent. By Theorem 9.6, this implies $\det \mathbf{A}^*\mathbf{A} = \mathrm{ddet}\mathbf{A} \neq 0$.

Sufficiency. Since $\mathrm{ddet}\mathbf{A} = \det \mathbf{A}^*\mathbf{A} \neq 0$, then by Theorem 7.1 there exists the inverse $(\mathbf{A}^*\mathbf{A})^{-1}$ of the Hermitian matrix $\mathbf{A}^*\mathbf{A}$. Multiplying it on the right by $\mathbf{A}^*$, we obtain the left inverse $(L\mathbf{A})^{-1} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*$. By representing $(\mathbf{A}^*\mathbf{A})^{-1} = (\frac{L_{ij}}{\mathrm{ddet}\mathbf{A}})_{n\times n}$ as the left inverse matrix, we get

$$(L\mathbf{A})^{-1} = (L(\mathbf{A}^*\mathbf{A}))^{-1}\mathbf{A}^* =$$

$$= \frac{1}{\mathrm{ddet}\mathbf{A}} \begin{pmatrix} \sum\limits_k L_{k1}a^*_{k1} & \sum\limits_k L_{k1}a^*_{k2} & \ldots & \sum\limits_k L_{k1}a^*_{kn} \\ \sum\limits_k L_{k2}a^*_{k1} & \sum\limits_k L_{k2}a^*_{k2} & \ldots & \sum\limits_k L_{k2}a^*_{kn} \\ \ldots & \ldots & \ldots & \ldots \\ \sum\limits_k L_{kn}a^*_{k1} & \sum\limits_k L_{kn}a^*_{k2} & \ldots & \sum\limits_k L_{kn}a^*_{kn} \end{pmatrix}$$

$$= \frac{1}{\mathrm{ddet}\mathbf{A}} \begin{pmatrix} \mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}\left(\mathbf{a}^*_{.1}\right) & \mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}\left(\mathbf{a}^*_{.2}\right) & \ldots & \mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.1}\left(\mathbf{a}^*_{.n}\right) \\ \mathrm{cdet}_2(\mathbf{A}^*\mathbf{A})_{.2}\left(\mathbf{a}^*_{.1}\right) & \mathrm{cdet}_2(\mathbf{A}^*\mathbf{A})_{.2}\left(\mathbf{a}^*_{.2}\right) & \ldots & \mathrm{cdet}_2(\mathbf{A}^*\mathbf{A})_{.2}\left(\mathbf{a}^*_{.n}\right) \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{cdet}_n(\mathbf{A}^*\mathbf{A})_{.n}\left(\mathbf{a}^*_{.1}\right) & \mathrm{cdet}_n(\mathbf{A}^*\mathbf{A})_{.n}\left(\mathbf{a}^*_{.2}\right) & \ldots & \mathrm{cdet}_n(\mathbf{A}^*\mathbf{A})_{.n}\left(\mathbf{a}^*_{.n}\right) \end{pmatrix}$$

By virtue of

$$\mathrm{ddet}\mathbf{A} = \det(\mathbf{A}^*\mathbf{A}) = \mathrm{cdet}_j(\mathbf{A}^*\mathbf{A}) = \sum_i \mathrm{cdet}_j(\mathbf{A}^*\mathbf{A})_{.j}\left(\mathbf{a}^*_{.i}\right) \cdot a_{ij} = \sum_i \mathbb{L}_{ij} \cdot a_{ij},$$

for all $j = \overline{1, n}$, we obtain (20).

Now we prove the formula (21). By Theorem 7.1 there exists an inverse matrix $(\mathbf{A}\mathbf{A}^*)^{-1} = (\frac{R_{ij}}{\mathrm{ddet}\mathbf{A}})_{n\times n}$. By having left-multiplied it by $\mathbf{A}^*$, we obtain

$$
\begin{aligned}
&(R\mathbf{A})^{-1} = \mathbf{A}^*\left(R\left(\mathbf{A}^*\mathbf{A}\right)\right)^{-1} = \\
&= \tfrac{1}{\mathrm{ddet}\mathbf{A}}
\begin{pmatrix}
\sum\limits_k a^*_{1k} R_{1k} & \sum\limits_k a^*_{1k} R_{2k} & \ldots & \sum\limits_k a^*_{1k} R_{nk} \\
\sum\limits_k a^*_{2k} R_{1k} & \sum\limits_k a^*_{2k} R_{1k} & \ldots & \sum\limits_k a^*_{2k} R_{nk} \\
\ldots & \ldots & \ldots & \ldots \\
\sum\limits_k a^*_{nk} R_{1k} & \sum\limits_k a^*_{nk} R_{2k} & \ldots & \sum\limits_k a^*_{nk} R_{nk}
\end{pmatrix} \\
&= \tfrac{1}{\mathrm{ddet}\mathbf{A}^*}
\begin{pmatrix}
\mathrm{rdet}_1(\mathbf{A}\mathbf{A}^*)_{1.}\left(\mathbf{a}^*_{1.}\right) & \mathrm{rdet}_2(\mathbf{A}\mathbf{A}^*)_{2.}\left(\mathbf{a}^*_{1.}\right) & \ldots & \mathrm{rdet}_n(\mathbf{A}\mathbf{A}^*)_{n.}\left(\mathbf{a}^*_{1.}\right) \\
\mathrm{rdet}_1(\mathbf{A}\mathbf{A}^*)_{1.}\left(\mathbf{a}^*_{2.}\right) & \mathrm{rdet}_2(\mathbf{A}\mathbf{A}^*)_{2.}\left(\mathbf{a}^*_{2.}\right) & \ldots & \mathrm{rdet}_n(\mathbf{A}\mathbf{A}^*)_{n.}\left(\mathbf{a}^*_{2.}\right) \\
\ldots & \ldots & \ldots & \ldots \\
\mathrm{rdet}_1(\mathbf{A}\mathbf{A}^*)_{1.}\left(\mathbf{a}^*_{n.}\right) & \mathrm{rdet}_2(\mathbf{A}\mathbf{A}^*)_{2.}\left(\mathbf{a}^*_{n.}\right) & \ldots & \mathrm{rdet}_n(\mathbf{A}\mathbf{A}^*)_{n.}\left(\mathbf{a}^*_{n.}\right)
\end{pmatrix}
\end{aligned}
$$

By virtue of

$$\mathrm{ddet}\mathbf{A} = \mathrm{rdet}_i(\mathbf{A}\mathbf{A}^*) = \sum_j a_{ij} \cdot \mathrm{rdet}_i(\mathbf{A}\mathbf{A}^*)_{i.}\left(\mathbf{a}^*_{j.}\right) = \sum_j a_{ij} \cdot \mathbb{R}_{ij},$$

for all $i = \overline{1, n}$, the formula (21) is valid. The equality $(L\mathbf{A})^{-1} = (R\mathbf{A})^{-1}$ is immediately from the well-known fact that if there exists an inverse matrix over an arbitrary skew field, then it is unique.■

Remark 11.1. *In Theorem 11.1, the inverse matrix* $\mathbf{A}^{-1}$ *of an arbitrary* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *under the assumption of* $\mathrm{ddet}\mathbf{A} \neq 0$ *is represented by the analog of the classical adjoint matrix. If we denote this analog of the adjoint matrix over* $\mathbb{H}$ *by* $\mathrm{Adj}[[\mathbf{A}]]$*, then the next formula is valid over* $\mathbb{H}$*:*

$$\mathbf{A}^{-1} = \frac{\mathrm{Adj}[[\mathbf{A}]]}{\mathrm{ddet}\mathbf{A}}.$$

Remark 11.2. *From Theorems 9.6 and 10.2 follows that for an arbitrary matrix* $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$ *the double determinant* $\mathrm{ddet}\mathbf{A}$ *satisfies Axioms 1, 2, 3 of Definition 1.1.*

12. The Relations between the Noncommutative Determinants

It is evident that $\mathrm{ddet}\mathbf{A} = \mathrm{Mdet}\,(\mathbf{A}^*\mathbf{A})$, then from [1, 6] we have the following relations between the noncommutative determinants of Moore, Study, Diedonné and the double determinant,

$$\mathrm{ddet}\mathbf{A} = \mathrm{Mdet}\,(\mathbf{A}^*\mathbf{A}) = \mathrm{Sdet}\mathbf{A} = \mathrm{Ddet}^2\mathbf{A}.$$

Due to wide use recently the quasideterminants of Gelfand-Retax relations between them and the row and column determinants can be important.

Theorem 12.1. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is a invertible matrix, then there are the following representations of a quasideterminant* $\mid \mathbf{A} \mid_{pq}$ *for all* $p, q = 1, ..., n$

$$\mid \mathbf{A} \mid_{pq} = \frac{\mathrm{ddet}\mathbf{A} \cdot \overline{\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.\,q}\left(\mathbf{a}^*_{.p}\right)}}{\mathrm{n}(\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.\,q}\left(\mathbf{a}^*_{.p}\right))}, \tag{22}$$

$$\mid \mathbf{A} \mid_{pq}= \frac{\mathrm{ddet}\mathbf{A} \cdot \overline{\mathrm{rdet}_p(\mathbf{A}\mathbf{A}^*)_{p.}\left(\mathbf{a}^*_{q.}\right)}}{\mathrm{n}(\mathrm{rdet}_p(\mathbf{A}\mathbf{A}^*)_{p.}\left(\mathbf{a}^*_{q.}\right))}. \tag{23}$$

Proof. Let $\mathbf{A}^{-1} = (b_{ij})$ is the inverse of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$. By (1) there is simple relationship between the quasideterminant $\mid \mathbf{A} \mid_{p,q}$ of $\mathbf{A} \in \mathrm{M}(n, \mathbf{H})$ and an element of its inverse $\mathbf{A}^{-1} = (b_{ij})$, that is

$$\mid \mathbf{A} \mid_{pq}= b_{qp}^{-1}$$

for all $p, q = 1, ..., n$. At the same time the theory of column and row determinants by Theorem (11.1) gives the deteminantal representations of the inverse matrix by the left (20) and right (21) double complements. So we have

$$\mid \mathbf{A} \mid_{pq}= b_{qp}^{-1} = \left(\frac{\mathbb{L}_{pq}}{\mathrm{ddet}\mathbf{A}}\right)^{-1} = \left(\frac{\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.q}\left(\mathbf{A}^*_{.p}\right)}{\mathrm{ddet}\mathbf{A}}\right)^{-1}, \tag{24}$$

$$\mid \mathbf{A} \mid_{pq}= b_{qp}^{-1} = \left(\frac{\mathbb{R}_{pq}}{\mathrm{ddet}\mathbf{A}}\right)^{-1} = \left(\frac{\mathrm{rdet}_p(\mathbf{A}\mathbf{A}^*)_{p.}\left(\mathbf{A}^*_{q.}\right)}{\mathrm{ddet}\mathbf{A}}\right)^{-1}. \tag{25}$$

Since $\mathrm{ddet}\mathbf{A} \neq 0 \in \mathbf{F}$, then $\exists (\mathrm{ddet}\mathbf{A})^{-1} \in \mathbf{F}$. In turn, we have

$$\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.q}\left(\mathbf{A}^*_{.p}\right)^{-1} = \frac{\overline{\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.q}\left(\mathbf{A}^*_{.p}\right)}}{\mathrm{n}(\mathrm{cdet}_q(\mathbf{A}^*\mathbf{A})_{.q}\left(\mathbf{A}^*_{.p}\right))}, \tag{26}$$

$$\mathrm{rdet}_p(\mathbf{A}\mathbf{A}^*)_{p.}\left(\mathbf{A}^*_{q.}\right)^{-1} = \frac{\overline{\mathrm{rdet}_p(\mathbf{A}\mathbf{A}^*)_{p.}\left(\mathbf{A}^*_{q.}\right)}}{\mathrm{n}(\mathrm{rdet}_p(\mathbf{A}\mathbf{A}^*)_{p.}\left(\mathbf{A}^*_{q.}\right))}. \tag{27}$$

Substituting (26) in (24) and (27) in (25), we respectively obtain (22) (23). ■

The formula (22) represent the quasideterminant $\mid \mathbf{A} \mid_{p,q}$ of $\mathbf{A} \in \mathrm{M}(n, \mathbf{H})$ for all $p, q = 1, ..., n$ by the column determinant of $\mathbf{A}^*\mathbf{A}$, and (23) represent the quasideterminant by the row determinant of $\mathbf{A}\mathbf{A}^*$.

13. Cramer's Rule for Systems of Linear Equations over Quaternion Algebra

Theorem 13.1. *Let*

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{y} \tag{28}$$

be a right system of linear equations with a matrix of coefficients $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, a column of constants* $\mathbf{y} = (y_1, \ldots, y_n)^T \in \mathbb{H}^{n\times 1}$*, and a column of unknowns* $\mathbf{x} = (x_1, \ldots, x_n)^T$*. If* $\mathrm{ddet}\mathbf{A} \neq 0$*, then the solution to the linear system (28) is given by components*

$$x_j = \frac{\mathrm{cdet}_j(\mathbf{A}^*\mathbf{A})_{.j}\left(\mathbf{f}\right)}{\mathrm{ddet}\mathbf{A}}, \quad j = \overline{1, n}, \tag{29}$$

where $\mathbf{f} = \mathbf{A}^*\mathbf{y}$.

Proof. By Theorem 11.1, $\mathbf{A}$ is invertibility. Thus, there exists the unique inverse matrix $\mathbf{A}^{-1}$. From this the existence and uniqueness of solutions of (28) follows immediately. Consider $\mathbf{A}^{-1}$ as the left inverse: $(L\mathbf{A})^{-1} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*$. Then we get

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{y} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^* \cdot \mathbf{y}.$$

Denote $\mathbf{f} := \mathbf{A}^* \cdot \mathbf{y}$. Here $\mathbf{f} = (f_1, f_2, ..., f_n)^T$ is the n-dimension column vector over $\mathbb{H}$. By considering $(\mathbf{A}^*\mathbf{A})^{-1}$ as the left inverse, the solution of (28) is represented by components:

$$x_j = (\mathrm{ddet}\mathbf{A})^{-1} \sum_{i=1}^{n} L_{ij} \cdot f_i, \ j = \overline{1, n},$$

where L_{ij} is the left ij-th cofactor of the Hermitian matrix $(\mathbf{A}^*\mathbf{A})$. From here we obtain (29).■

Theorem 13.2. *Let*

$$\mathbf{x} \cdot \mathbf{A} = \mathbf{y} \tag{30}$$

be a left system of linear equations with a matrix of coefficients $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, a row of constants* $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{H}^{1\times n}$*, and a row of unknowns* $\mathbf{x} = (x_1, \ldots, x_n)$*. If* $\mathrm{ddet}\mathbf{A} \neq 0$*, then the solution to the linear system (30) is given by components*

$$x_i = \frac{\mathrm{rdet}_i(\mathbf{A}\mathbf{A}^*)_{i.}\,(\mathbf{z})}{\mathrm{ddet}\mathbf{A}}, \quad i = \overline{1, n} \tag{31}$$

where $\mathbf{z} = \mathbf{y}\mathbf{A}^*$*.*

Proof is similar to the proof of Theorem 13.1.

Remark 13.1. *The formulas (29) and (31) are the obvious and natural generalizations of Cramer's rule for systems of linear equations over quaternion algebra.*

The closest analog to Cramer's rule, as follows from Theorem 7.1, can be obtained in the following specific cases.

Theorem 13.3. *If the matrix of coefficients* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *in the right system of linear equations over* $\mathbb{H}$ *(28) is Hermitian, then the unique solution vector* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *of the system is given by*

$$x_j = \frac{\mathrm{cdet}_j\mathbf{A}_{.j}\,(\mathbf{y})}{\det \mathbf{A}} \quad j = \overline{1, n}.$$

Theorem 13.4. *If the matrix of coefficients* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *in the left system of linear equations over* $\mathbb{H}$ *(30) is Hermitian, then the unique solution vector* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *is given by*

$$x_i = \frac{\mathrm{rdet}_i\mathbf{A}_{i.}\,(\mathbf{y})}{\det \mathbf{A}} \quad i = \overline{1, n}.$$

14. Cramer's Rule for Some Matrix Equations

We denote $\mathbf{A}^*\mathbf{B} =: \hat{\mathbf{B}} = (\hat{b}_{ij})$, $\mathbf{B}\mathbf{A}^* =: \check{\mathbf{B}} = (\check{b}_{ij})$.

Theorem 14.1. *Suppose*

$$\mathbf{A}\mathbf{X} = \mathbf{B} \tag{32}$$

is a right matrix equation, where $\{\mathbf{A}, \mathbf{B}\} \in \mathrm{M}(n, \mathbb{H})$ *are given,* $\mathbf{X} \in \mathrm{M}(n, \mathbb{H})$ *is unknown. If* $\mathrm{ddet}\mathbf{A} \neq 0$*, then (32) has a unique solution, and the solution is*

$$x_{i\,j} = \frac{\mathrm{cdet}_i(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\hat{\mathbf{b}}_{.j}\right)}{\mathrm{ddet}\mathbf{A}} \tag{33}$$

where $\hat{\mathbf{b}}_{.j}$ *is the* j*th column of* $\hat{\mathbf{B}}$ *for all* $i, j = 1, ..., n$*.*

Proof. By Theorem 11.1 the matrix $\mathbf{A}$ is invertible. There exists the unique inverse matrix $\mathbf{A}^{-1}$. From this it follows that the solution of (32) exists and is unique, $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. If we represent $\mathbf{A}^{-1} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*$ as a left inverse and use the determinantal representation of $(\mathbf{A}^*\mathbf{A})^{-1}$ by (13), then for all $i, j = 1, ..., n$ we obtain

$$x_{ij} = \frac{1}{\mathrm{ddet}\mathbf{A}} \sum_{k=1}^{n} L_{ki}\hat{b}_{kj},$$

where L_{ij} is a left ijth cofactor of $(\mathbf{A}^*\mathbf{A})$ for all $i, j = 1, ..., n.$ From this by Lemma 3.1 and denoting the j-th column of $\hat{\mathbf{B}}$ by $\hat{\mathbf{b}}_{.j}$, it follows (33).■

Theorem 14.2. *Suppose*

$$\mathbf{X}\mathbf{A} = \mathbf{B} \tag{34}$$

is a left matrix equation, where $\{\mathbf{A}, \mathbf{B}\} \in \mathrm{M}(n, \mathbb{H})$ *are given,* $\mathbf{X} \in \mathrm{M}(n, \mathbb{H})$ *is unknown. If* $\mathrm{ddet}\mathbf{A} \neq 0$*, then (8) has a unique solution, and the solution is*

$$x_{i\,j} = \frac{\mathrm{rdet}_j(\mathbf{A}\mathbf{A}^*)_{j.}\left(\check{\mathbf{b}}_{i\,.}\right)}{\mathrm{ddet}\mathbf{A}} \tag{35}$$

where $\check{\mathbf{b}}_{i.}$ *is the* i*th column of* $\check{\mathbf{B}}$ *for all* $i, j = 1, ..., n$*.*

Proof. By Theorem 11.1 the matrix $\mathbf{A}$ is invertible. There exists the unique inverse matrix $\mathbf{A}^{-1}$. From this it follows that the solution of (34) exists and is unique, $\mathbf{X} = \mathbf{B}\mathbf{A}^{-1}$. If we represent $(\mathbf{A})^{-1} = \mathbf{A}^*\left(\mathbf{A}\mathbf{A}^*\right)^{-1}$ as a right inverse and use the determinantal representation of $(\mathbf{A}\mathbf{A}^*)^{-1}$ by (12), then for all $i, j = 1, ..., n$ we have

$$x_{ij} = \frac{1}{\mathrm{ddet}\mathbf{A}} \sum_{k=1}^{n} \check{b}_{i\,k}R_{jk}.$$

where $R_{i\,j}$ is a right ijth cofactor of $(\mathbf{A}\mathbf{A}^*)$ for all $i, j = 1, ..., n.$ From this by means of Lemma 3.2 and denoting the ith row of $\check{\mathbf{B}}$ by $\check{\mathbf{b}}_{i\,.}$, it follows (35). ■

We denote $\mathbf{A}^*\mathbf{C}\mathbf{B}^* =: \tilde{\mathbf{C}} = (\tilde{c}_{ij})$.

Theorem 14.3. *Suppose*

$$\mathbf{AXB} = \mathbf{C} \tag{36}$$

is a two-sided matrix equation, where $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\} \in \mathrm{M}(n, \mathbb{H})$ *are given,* $\mathbf{X} \in \mathrm{M}(n, \mathbb{H})$ *is unknown. If* $\mathrm{ddet}\mathbf{A} \neq 0$ *and* $\mathrm{ddet}\mathbf{B} \neq 0$ *, then (36) has a unique solution, and the solution is*

$$x_{i\,j} = \frac{\mathrm{rdet}_j(\mathbf{BB}^*)_{j.}\,\left(\mathbf{c}^{\mathbf{A}}_{i\,.}\right)}{\mathrm{ddet}\mathbf{A}\cdot\mathrm{ddet}\mathbf{B}}, \tag{37}$$

or

$$x_{i\,j} = \frac{\mathrm{cdet}_i(\mathbf{A}^*\mathbf{A})_{.\,i}\,\left(\mathbf{c}^{\mathbf{B}}_{.j}\right)}{\mathrm{ddet}\mathbf{A}\cdot\mathrm{ddet}\mathbf{B}}, \tag{38}$$

where $\mathbf{c}^{\mathbf{A}}_{i\,.} := \left(\mathrm{cdet}_i(\mathbf{A}^*\mathbf{A})_{.\,i}\,(\tilde{\mathbf{c}}_{.1}),\ldots,\mathrm{cdet}_i(\mathbf{A}^*\mathbf{A})_{.\,i}\,(\tilde{\mathbf{c}}_{.n})\right)$ *is the row vector and* $\mathbf{c}^{\mathbf{B}}_{.j} := \left(\mathrm{rdet}_j(\mathbf{BB}^*)_{j.}\,(\tilde{\mathbf{c}}_{1\,.}),\ldots,\mathrm{rdet}_j(\mathbf{BB}^*)_{j.}\,(\tilde{\mathbf{c}}_{\underset{.}{n}\,.})\right)^T$ *is the column vector and* $\tilde{\mathbf{c}}_{i\,.}$, $\tilde{\mathbf{c}}_{.\,j}$ *are the ith row vector and the jth column vector of* $\widetilde{\mathbf{C}}$*, respectively, for all* $i, j = 1, ..., n$.

Proof. By Theorem 11.1 the matrices $\mathbf{A}$ and $\mathbf{B}$ are invertible. There exist the unique inverse matrices $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$. From this it follows that the solution of (36) exists and is unique, $\mathbf{X} = \mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1}$. If we represent $\mathbf{A}^{-1} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*$ as a left inverse and $(\mathbf{B})^{-1} = \mathbf{B}^*\,(\mathbf{BB}^*)^{-1}$ as a right inverse, then for all $i, j = 1, ..., n$ we have

$$\mathbf{X} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*\mathbf{C}\mathbf{B}^*\,(\mathbf{BB}^*)^{-1} =$$

$$= \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{pmatrix} = \frac{1}{\mathrm{ddet}\mathbf{A}} \begin{pmatrix} L^{\mathbf{A}}_{11} & L^{\mathbf{A}}_{21} & \ldots & L^{\mathbf{A}}_{n1} \\ L^{\mathbf{A}}_{12} & L^{\mathbf{A}}_{22} & \ldots & L^{\mathbf{A}}_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ L^{\mathbf{A}}_{1n} & L^{\mathbf{A}}_{2n} & \ldots & L^{\mathbf{A}}_{nn} \end{pmatrix} \times$$

$$\times \begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \ldots & \tilde{c}_{1n} \\ \tilde{c}_{21} & \tilde{c}_{22} & \ldots & \tilde{c}_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \tilde{c}_{n1} & \tilde{c}_{n2} & \ldots & \tilde{c}_{nn} \end{pmatrix} \frac{1}{\mathrm{ddet}\mathbf{A}} \begin{pmatrix} R^{\mathbf{B}}_{11} & R^{\mathbf{B}}_{21} & \ldots & R^{\mathbf{B}}_{n1} \\ R^{\mathbf{B}}_{12} & R^{\mathbf{B}}_{22} & \ldots & R^{\mathbf{B}}_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ R^{\mathbf{B}}_{1n} & R^{\mathbf{B}}_{2n} & \ldots & R^{\mathbf{B}}_{nn} \end{pmatrix},$$

where $L^{\mathbf{A}}_{ij}$ is a left ijth cofactor of $(\mathbf{A}^*\mathbf{A})$ and $R^{\mathbf{B}}_{ij}$ is a right ijth cofactor of $(\mathbf{BB}^*)$ for all $i, j = 1, ..., n$. This implies

$$x_{ij} = \frac{\sum\limits_{m=1}^{n}\left(\sum\limits_{k=1}^{n} L^{\mathbf{A}}_{ki}\tilde{c}_{\,km}\right) R^{\mathbf{B}}_{jm}}{\mathrm{ddet}\mathbf{A}\cdot\mathrm{ddet}\mathbf{B}}, \tag{39}$$

for all $i, j = \overline{1, n}$. From this by Lemma 3.2, we obtain

$$\sum_{k=1}^{n} L^{\mathbf{A}}_{ki}\tilde{c}_{k\,m} = \mathrm{cdet}_i(\mathbf{A}^*\mathbf{A})_{.\,i}\,(\tilde{\mathbf{c}}_{.\,m}),$$

where $\tilde{\mathbf{c}}_{.\,m}$ is the mth column-vector of $\tilde{\mathbf{C}}$ for all $m = 1, ..., n$. Denote by $\mathbf{c}^{\mathbf{A}}_{i\,.} := \left(\mathrm{cdet}_i(\mathbf{A}^*\mathbf{A})_{.\,i}\,(\tilde{\mathbf{c}}_{.1}),\ldots,\mathrm{cdet}_i(\mathbf{A}^*\mathbf{A})_{.\,i}\,(\tilde{\mathbf{c}}_{.n})\right)$ the row-vector for all $i = 1, ..., n$. Reducing the sum $\sum\limits_{m=1}^{n}\left(\sum\limits_{k=1}^{n} L^{\mathbf{A}}_{ki}\tilde{c}_{\,km}\right) R^{\mathbf{B}}_{jm}$ by Lemma 3.1, we obtain an analog of Cramer's rule for (36) by (37).

Having changed the order of summation in (39), we have

$$x_{ij} = \frac{\sum\limits_{k=1}^{n} L^{\mathbf{A}}_{ki} \left(\sum\limits_{m=1}^{n} \tilde{c}_{km} R^{\mathbf{B}}_{jm} \right)}{\mathrm{ddet}\mathbf{A} \cdot \mathrm{ddet}\mathbf{B}}.$$

By Lemma 3.1, we obtain $\sum\limits_{m=1}^{n} c_{k\,m} R^{\mathbf{B}}_{j\,m} = \mathrm{rdet}_j(\mathbf{BB}^*)_{j.}\,(\tilde{\mathbf{c}}_{k\,.})$, where $\tilde{\mathbf{c}}_{k\,.}$ is a kth row-vector of $\tilde{\mathbf{C}}$ for all $k = 1, ..., n$. We denote by

$$\mathbf{c}^{\mathbf{B}}_{.\,j} := \left(\mathrm{rdet}_j(\mathbf{BB}^*)_{j.}\,(\tilde{\mathbf{c}}_{1\,.}), \ldots, \mathrm{rdet}_j(\mathbf{BB}^*)_{j.}\,(\tilde{\mathbf{c}}_{n\,.})\right)^T$$

the column-vector for all $j = 1, ..., n$. Reducing the sum $\sum\limits_{k=1}^{n} L^{\mathbf{A}}_{ki} \left(\sum\limits_{m=1}^{n} \tilde{c}_{km} R^{\mathbf{B}}_{jm} \right)$ by Lemma 3.2, we obtain Cramer's rule for (36) by (38). ■

In solving the matrix equations by Cramer's rules (33), (35), (37), (38) we do not use the complex representation of quaternion matrices and work only in the quaternion algebra.

15. Example 1

Let us consider the two-sided matrix equation

$$\mathbf{AXB} = \mathbf{C} \tag{40}$$

where

$$\mathbf{A} = \begin{pmatrix} i & -j & k \\ k & -i & 1 \\ 2 & k & -j \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -k & j & 2 \\ i & k & i \\ -j & 1 & i \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 1 & i & j \\ k & j & -2 \\ i & 1 & j \end{pmatrix}.$$

Then we have

$$\mathbf{A}^* = \begin{pmatrix} -i & -k & 2 \\ j & i & -k \\ -k & 1 & j \end{pmatrix}, \ \mathbf{A}^*\mathbf{A} = \begin{pmatrix} 6 & j+3k & -j-k \\ -j-3k & 3 & i \\ j+k & -i & 3 \end{pmatrix}$$

and

$$\mathbf{B}^* = \begin{pmatrix} k & -i & j \\ -j & -k & 1 \\ 2 & -i & -i \end{pmatrix}, \mathbf{BB}^* = \begin{pmatrix} 6 & -3i+j & -i+j \\ 3i-j & 3 & 1+2k \\ i-j & 1-2k & 3 \end{pmatrix},$$

$$\tilde{\mathbf{C}} = \mathbf{A}^*\mathbf{C}\mathbf{B}^* = \begin{pmatrix} 2k & 1-i-k & 3+i+3k \\ -2-4i & -2+i-k & i-k \\ -4+2i & 1+2i+j & 1+4i+j \end{pmatrix}.$$

It is easy to get, $\mathrm{ddet}\mathbf{A} = \det \mathbf{A}^*\mathbf{A} = 8$ and $\mathrm{ddet}\mathbf{B} = \det \mathbf{BB}^* = 4$. Therefore (40) has a solution. We shall find it by (37). At first we obtain the row-vectors $\mathbf{c}^{\mathbf{A}}_{i\,.}$ for all $i = 1, 2, 3$.

$$\begin{array}{l}
\mathrm{cdet}_1(\mathbf{A}^*\mathbf{A})_{.\,1}\,(\tilde{\mathbf{c}}_{.1}) = \mathrm{cdet}_1 \begin{pmatrix} 2k & j+3k & -j-k \\ -2-4i & 3 & i \\ -4+2i & -i & 3 \end{pmatrix} = 3\cdot 3(2k)- \\
-i(-i)(3j+5k) + (-j-k)(-i)(-2-4i) - 3(j+3k)(-2-4i)+ \\
+(j+3k)i(-4+2i) - 3(-j-k)(-4+2i) = \\
= 24j + 8k,
\end{array}$$

and so forth. Continuing in the same way, we get

$$\begin{array}{l}
\mathbf{c}^{\mathbf{A}}_{1\,.} = (24j+8k, -8-8i+4j+4k, 8+8i+4j+4k), \\
\mathbf{c}^{\mathbf{A}}_{2\,.} = (-20-36i, -10-2i-12j-12k, -2-2i+12j+4k), \\
\mathbf{c}^{\mathbf{A}}_{3\,.} = (12+4i, 6+2i+12j-4k, 6+10i-4j+4k).
\end{array}$$

Then by (37) we have

$$\begin{array}{l}
x_{1\,1} = \frac{\mathrm{rdet}_1(\mathbf{B}\mathbf{B}^*)_{1.}\left(\mathbf{c}^{\mathbf{A}}_{1\,.}\right)}{\mathrm{ddet}\mathbf{A}\cdot\mathrm{ddet}\mathbf{B}} = \\
= \frac{1}{32}\cdot\mathrm{rdet}_1 \begin{pmatrix} 24j+8k & -8-8i+4j+4k & 8+8i+4j+4k \\ 3i-j & 3 & 1+2k \\ i-j & 1-2k & 3 \end{pmatrix} = \\
= \frac{1}{30}\cdot((24j+8k)\cdot 3\cdot 3 - (24j+8k)(1+2k)(1-2k)+ \\
+(-8-8i+4j+4k)(1+2k)(i-j) - (-8-8i+4j+4k)(3i-j)3+ \\
+(8+8i+4j+4k)(1-2k)(3i-j) - (8+8i+4j+4k)(i-j)3) = \\
= \frac{1}{32}\cdot(-32+32i),
\end{array}$$

and so forth. Continuing in the same way, we obtain

$$\begin{array}{rrr}
x_{11} = \frac{-32+32i}{32}, & x_{12} = \frac{-88-72i+24j-8k}{32} & x_{13} = \frac{24+8i-40j+56k}{32}, \\
x_{21} = \frac{-16i+32j-48k}{32}, & x_{22} = \frac{20-28i-116j-76k}{32}, & x_{23} = \frac{-44+68i+20j+12k}{32}, \\
x_{31} = \frac{16+16j+32k}{32}, & x_{32} = \frac{20+44i+52j-28k}{32}, & x_{32} = \frac{-12-20i+12j-4k}{32}.
\end{array}$$

16. The Singular Value Decomposition and the Moore-Penrose Inverse of a Quaternion Matrix

In the all following sections we shall consider the Hamilton quaternion skew field $\mathbb{H}$ (the quaternion algebra over the real field).

Due to the noncommutativity of quaternions, there are two types of eigenvalues.

Definition 16.1. *Let* $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$. *A quaternion* λ *is said to be a right eigenvalue of* $\mathbf{A}$ *if* $\mathbf{A}\cdot\mathbf{x} = \mathbf{x}\cdot\lambda$ *for some nonzero quaternion column-vector* $\mathbf{x}$. *Similarly* λ *is a left eigenvalue if* $\mathbf{A}\cdot\mathbf{x} = \lambda\cdot\mathbf{x}$.

The theory on the left eigenvalues of quaternion matrices has been investigated in particular in [12, 24, 26]. The theory on the right eigenvalues of quaternion matrices is more developed. In particular we note [2, 8, 27]. From this theory we cite the following propositions.

Proposition 16..1. *[27] Let* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian. Then* $\mathbf{A}$ *has exactly* n *real right eigenvalues.*

Definition 16.2. *Suppose* $\mathbf{U} \in \mathrm{M}(n, \mathbb{H})$ *and* $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I}$*, then the matrix* $\mathbf{U}$ *is called unitary.*

Proposition 16..2. *[27] Let* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *be given. Then,* $\mathbf{A}$ *is Hermitian if and only if there are a unitary matrix* $\mathbf{U} \in \mathrm{M}(n, \mathbb{H})$ *and a real diagonal matrix* $\mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *such that* $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^*$*, where* $\lambda_1, ..., \lambda_n$ *are right eigenvalues of* $\mathbf{A}$.

Suppose $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is Hermitian and $\lambda \in \mathbb{R}$ is its right eigenvalue, then $\mathbf{A} \cdot \mathbf{x} = \mathbf{x} \cdot \lambda = \lambda \cdot \mathbf{x}$. This means that all right eigenvalues of a Hermitian matrix are its left eigenvalues as well. For real left eigenvalues, $\lambda \in \mathbb{R}$, the matrix $\lambda\mathbf{I} - \mathbf{A}$ is Hermitian.

Definition 16.3. *If* $t \in \mathbb{R}$*, then for a Hermitian matrix* $\mathbf{A}$ *the polynomial* $p_{\mathbf{A}}(t) = \det(t\mathbf{I} - \mathbf{A})$ *is said to be the characteristic polynomial of* $\mathbf{A}$.

The roots of the characteristic polynomial of a Hermitian matrix are its real left eigenvalues, which are its right eigenvalues as well. We shall investigate coefficients of the characteristic polynomial like to the commutative case (see, e.g. [21]). At first we prove the auxiliary lemma.

Lemma 16.1. *Let* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian and the columns* $i_1, \ldots, i_k$ *of* $\mathbf{A}$ *coincide with the unit vectors* $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}$*. Then* $\det \mathbf{A}$ *equals to a principal minor obtained from* $\mathbf{A}$ *by deleting the rows and columns* $i_1, \ldots, i_k$.

Proof. We claim that if $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is Hermitian and the columns $i_1, \ldots, i_k$ of $\mathbf{A}$ coincide with the unit column vectors $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}$ respectively, then the rows $i_1, \ldots, i_k$ coincide with the unit row vectors $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}$ as well. Using Lemma 3.2 we expand $\det \mathbf{A}$ along the i_1th column, where $a_{i_1 k} = 0$ for all $k \neq i_1$ and $a_{i_1 i_1} = 1$. Then we obtain

$$\begin{gathered}
\det \mathbf{A} = \mathrm{cdet}_{i_1} \mathbf{A} = \\
= -\mathrm{cdet}_{i_1} \mathbf{A}^{11}_{i_1 .}(\mathbf{a}_{1.}) \cdot a_{1 i_1} - \ldots + \mathrm{cdet}_1 \mathbf{A}^{i_1 i_1} \cdot a_{i_1 i_1} - \ldots - \mathrm{cdet}_{i_1} \mathbf{A}^{nn}_{i_1 .}(\mathbf{a}_{n.}) \cdot a_{n i_1} \\
= -\mathrm{cdet}_{i_1} \mathbf{A}^{11}_{i_1 .}(\mathbf{a}_{1.}) \cdot 0 + \ldots + \mathrm{cdet}_1 \mathbf{A}^{i_1 i_1} \cdot 1 + \ldots - \mathrm{cdet}_{i_1} \mathbf{A}^{nn}_{i_1 .}(\mathbf{a}_{n.}) \cdot 0 \\
= \mathrm{cdet}_1 \mathbf{A}^{i_1 i_1}.
\end{gathered}$$

Since the submatrix $\mathbf{A}^{i_1 i_1}$ is obtained from $\mathbf{A}$ by deleting both the i_1-th rows and columns, then by Theorem 4.1 it follows that $\mathrm{cdet}_1 \mathbf{A}^{i_1 i_1} = \det \mathbf{A}^{i_1 i_1}$. Now we calculate this principal minor expanding along the i_2-th column. Similarly to above we have that $\det \mathbf{A}$ is equals to a principal minor obtained from $\mathbf{A}$ by deleting both the i_1th and i_2th rows and columns. Continuing this line of reasoning we complete the proof of the lemma. ■

Taking into account Lemma 16.1 we can prove the following theorem by analogy to the commutative case (see, e.g. [21]).

Theorem 16.1. *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian, then* $p_{\mathbf{A}}(t) = t^n - d_1 t^{n-1} + d_2 t^{n-2} - \ldots + (-1)^n d_n$*, where* d_r *is the sum of principle minors of* $\mathbf{A}$ *of order* r*,* $1 \leq r < n$*, and* $d_n = \det \mathbf{A}$.

For any quaternion matrix $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$, the eigenvalues of $\mathbf{A}^*\mathbf{A}$ are all nonnegative real numbers [25].

Definition 16.4. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$. *The nonnegative square roots of the* n *eigenvalues of* $\mathbf{A}^*\mathbf{A}$ *are called the singular values of* $\mathbf{A}$.

A key value for a determinantal representation of the Moore-Penrose inverse over the quaternion skew field have the following singular value decomposition (SVD) theorem.

Theorem 16.2. *[25, 27] (SVD) Let* $\mathbf{A} \in \mathbb{H}_r^{m\times n}$. *Then there exist unitary quaternion matrices* $\mathbf{U}_1 \in \mathbb{H}^{m\times m}$ *and* $\mathbf{U}_2 \in \mathbb{H}^{n\times n}$ *such that*

$$\mathbf{U}_1\mathbf{A}\mathbf{U}_2 = \begin{bmatrix} \mathbf{D}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{H}^{m\times n}, \tag{41}$$

where $\mathbf{D}_r = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r), \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$, *and* $\sigma_1, \sigma_2, \ldots, \sigma_r$ *are the all nonzero singular values of* $\mathbf{A}$.

As unitary matrices are invertible, the equality (41) can be written as follows

$$\mathbf{A} = \mathbf{V}\mathbf{\Sigma}\mathbf{W}^*, \tag{42}$$

where $\mathbf{V} \in \mathbb{H}^{m\times m}$ and $\mathbf{W} \in \mathbb{H}^{n\times n}$ are unitary matrices, and the matrix $\mathbf{\Sigma} = (\sigma_{ij}) \in \mathbb{H}_r^{m\times n}$ is such that $\sigma_{11} \geq \sigma_{22} \geq \ldots \geq \sigma_{rr} > \sigma_{r+1\,r+1} = \ldots = \sigma_{qq} = 0$, $q = \min\{n, m\}$.

We get the following lemmas, which have the analogues in the complex case [11].

Lemma 16.2. *Suppose* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *has the singular value decomposition,* $\mathbf{A} = \mathbf{V}\mathbf{\Sigma}\mathbf{W}^*$. *Let* $\mathbf{A}^+ = \mathbf{W}\cdot\mathbf{\Sigma}^+\cdot\mathbf{V}^*$, *where* $\mathbf{\Sigma}^+ \in \mathbb{H}^{n\times m}$ *is obtained from* $\mathbf{\Sigma}$ *by transposition and replacing positive entries of* $\mathbf{\Sigma}$ *by reciprocal. Then for* $\mathbf{A}^+$ *the following conditions are true*

$$\begin{array}{l} 1)\ (\mathbf{A}\mathbf{A}^+)^* = \mathbf{A}\mathbf{A}^+; \\ 2)\ (\mathbf{A}^+\mathbf{A})^* = \mathbf{A}^+\mathbf{A}; \\ 3)\ \mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}; \\ 4)\ \mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+. \end{array} \tag{43}$$

Proof. We obviously have $\left(\mathbf{\Sigma}^T\right)^* = \mathbf{\Sigma}$ and $\left(\left(\mathbf{\Sigma}^+\right)^T\right)^* = \mathbf{\Sigma}^+$ for $\mathbf{\Sigma}$ from the SVD by (42) and $\mathbf{\Sigma}^+$. Then it follows that

$$(\mathbf{A}\mathbf{A}^+)^* = (\mathbf{V}\mathbf{\Sigma}\mathbf{W}^*\mathbf{W}\mathbf{\Sigma}^+\mathbf{V}^*)^* = (\mathbf{V}\mathbf{\Sigma}\mathbf{I}\mathbf{\Sigma}^+\mathbf{V}^*)^* = \left(\mathbf{V}\left(\mathbf{\Sigma}^+\right)^T\mathbf{\Sigma}^T\mathbf{V}^*\right)^* =$$
$$= \left(\mathbf{V}\left(\mathbf{\Sigma}^+\right)^T\mathbf{W}^*\mathbf{W}\mathbf{\Sigma}^T\mathbf{V}^*\right)^* = \mathbf{V}\mathbf{\Sigma}\mathbf{W}^*\mathbf{W}\mathbf{\Sigma}^+\mathbf{V}^* = \mathbf{A}\mathbf{A}^+.$$

The proof of 1) is completed. By analogy we can prove 2).

Now we prove the condition 3). Note that $\mathbf{\Sigma}\mathbf{\Sigma}^+ = \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{H}^{m\times m}$. This implies $\mathbf{\Sigma}\mathbf{\Sigma}^+\mathbf{\Sigma} = \mathbf{\Sigma}$, then $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{V}\mathbf{\Sigma}\mathbf{W}^*\cdot\mathbf{W}\mathbf{\Sigma}^+\mathbf{V}^*\cdot\mathbf{V}\mathbf{\Sigma}\mathbf{W}^* = \mathbf{V}\cdot\mathbf{\Sigma}\mathbf{\Sigma}^+\mathbf{\Sigma}\cdot\mathbf{W}^* = \mathbf{V}\cdot\mathbf{\Sigma}\cdot\mathbf{W}^* = \mathbf{A}$.

By analogy to 3) can be prove the condition 4).

■

Lemma 16.3. *There exists a unique matrix $\mathbf{A}^{+}$ that satisfies conditions 1)-4) in (43).*

Proof. Suppose that both matrices $\mathbf{B}\in\mathbb{H}^{n\times m}$ and $\mathbf{C}\in\mathbb{H}^{n\times m}$ satisfy conditions 1)-4) in Lemma 16.2. Then we have

$$\begin{aligned}&\mathbf{B}=\mathbf{BAB}=\mathbf{B}\left(\mathbf{AB}\right)^{*}=\mathbf{BB}^{*}\mathbf{A}^{*}=\mathbf{BB}^{*}\left(\mathbf{ACA}\right)^{*}=\mathbf{BB}^{*}\mathbf{A}^{*}\mathbf{C}^{*}\mathbf{A}^{*}\\&=\mathbf{B}\left(\mathbf{AB}\right)^{*}\left(\mathbf{AC}\right)^{*}=\mathbf{BABAC}=\mathbf{BAC}=\mathbf{BACAC}=\left(\mathbf{BA}\right)^{*}\left(\mathbf{CA}\right)^{*}\mathbf{C}\\&=\mathbf{A}^{*}\mathbf{B}^{*}\mathbf{A}^{*}\mathbf{C}^{*}\mathbf{C}=\left(\mathbf{ABA}\right)^{*}\mathbf{C}^{*}\mathbf{C}=\mathbf{A}^{*}\mathbf{C}^{*}\mathbf{C}=\left(\mathbf{CA}\right)^{*}\mathbf{C}=\mathbf{CAC}=\mathbf{C}.\end{aligned}$$

■

Definition 16.5. *Let $\mathbf{A}\in\mathbb{H}^{m\times n}$. The matrix $\mathbf{A}^{+}$ is called the Moore-Penrose inverse if it satisfies all conditions in (43).*

By analogy to the complex case [3] we have the theorem about the limit representation of the Moore-Penrose inverse.

Theorem 16.3. *If $\mathbf{A}\in\mathbb{H}^{m\times n}$ and $\mathbf{A}^{+}$ is its Moore-Penrose inverse, then $\mathbf{A}^{+}=\lim\limits_{\alpha\to 0}\mathbf{A}^{*}\left(\mathbf{AA}^{*}+\alpha\mathbf{I}\right)^{-1}=\lim\limits_{\alpha\to 0}\left(\mathbf{A}^{*}\mathbf{A}+\alpha\mathbf{I}\right)^{-1}\mathbf{A}^{*}$, where $\alpha\in\mathbb{R}_{+}$.*

Proof. Suppose $\mathbf{A}=\mathbf{V\Sigma W}^{*}$, then $\mathbf{A}^{*}=\mathbf{W\Sigma}^{*}\mathbf{V}^{*}$ and $\mathbf{A}^{+}=\mathbf{W\Sigma}^{+}\mathbf{V}^{*}$. Since $\mathbf{V}$ is unitary, then $\mathbf{V}^{*}=\mathbf{V}^{-1}$. We have

$$\begin{aligned}&\mathbf{A}^{*}\left(\mathbf{AA}^{*}+\alpha\mathbf{I}\right)^{-1}=\mathbf{W\Sigma V}^{*}\cdot\left(\mathbf{V\Sigma}\cdot\mathbf{W}^{*}\mathbf{W}\cdot\mathbf{\Sigma V}^{*}+\alpha\mathbf{I}\right)^{-1}=\\&=\mathbf{W\Sigma V}^{*}\cdot\left(\mathbf{V}\left(\mathbf{\Sigma\Sigma}^{*}+\alpha\mathbf{I}\right)\mathbf{V}^{*}\right)^{-1}=\mathbf{W\Sigma}\left(\mathbf{\Sigma\Sigma}^{*}+\alpha\mathbf{I}\right)^{-1}\mathbf{V}^{*}.\end{aligned}$$

Consider the matrix

$$\mathbf{\Sigma}\left(\mathbf{\Sigma\Sigma}^{*}+\alpha\mathbf{I}\right)^{-1}=\begin{pmatrix}\frac{\lambda_1}{\lambda_1^2+\alpha} & \dots & 0 & \\ \dots & \dots & \dots & \mathbf{0}\\ 0 & \dots & \frac{\lambda_r}{\lambda_r^2+\alpha} & \vdots\\ & \vdots & & \ddots\\ & \mathbf{0} & & \mathbf{0}\end{pmatrix}.$$

It is obviously that $\lim\limits_{\alpha\to 0}\mathbf{\Sigma}\left(\mathbf{\Sigma\Sigma}^{*}+\alpha\mathbf{I}\right)^{-1}=\mathbf{\Sigma}^{+}$. This implies $\lim\limits_{\alpha\to 0}\mathbf{A}^{*}\left(\mathbf{AA}^{*}+\alpha\mathbf{I}\right)^{-1}=\lim\limits_{\alpha\to 0}\mathbf{W\Sigma}\left(\mathbf{\Sigma\Sigma}^{*}+\alpha\mathbf{I}\right)^{-1}\mathbf{V}^{*}=\mathbf{A}^{+}$.

By analogy we can prove that $\mathbf{A}^{+}=\lim\limits_{\alpha\to 0}\left(\mathbf{A}^{*}\mathbf{A}+\alpha\mathbf{I}\right)^{-1}\mathbf{A}^{*}$. ■

It is evidently the following corollary.

Corollary 16.1. *If $\mathbf{A}\in\mathbb{H}^{m\times n}$, then the following statements are true.*

i) *If* $\operatorname{rank}\mathbf{A}=\mathrm{n}$, *then* $\mathbf{A}^{+}=\left(\mathbf{A}^{*}\mathbf{A}\right)^{-1}\mathbf{A}^{*}$.

ii) *If* $\operatorname{rank}\mathbf{A}=\mathrm{m}$, *then* $\mathbf{A}^{+}=\mathbf{A}^{*}\left(\mathbf{AA}^{*}\right)^{-1}$.

iii) *If* $\operatorname{rank}\mathbf{A}=\mathrm{n}=\mathrm{m}$, *then* $\mathbf{A}^{+}=\mathbf{A}^{-1}$.

17. Determinantal Representation of the Moore-Penrose Inverse

Lemma 17.1. *If* $\mathbf{A} \in \mathbb{H}_r^{m\times n}$*, then* $\operatorname{rank}\,(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right) \leq r$.

Proof. Let's lead elementary transformations of the matrix $(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)$ right-multiplying it by elementary unimodular matrices $\mathbf{P}_{i\,k}\left(-a_{jk}\right)$, $k \neq j$. The matrix $\mathbf{P}_{i\,k}\left(-a_{jk}\right)$ has $-a_{j\,k}$ in the (i,k) entry, 1 in all diagonal entries, and 0 in others. It is the matrix of an elementary transformation. Right-multiplying a matrix by $\mathbf{P}_{i\,k}\left(-a_{jk}\right)$, where $k \neq j$, means adding to kth column its ith column right-multiplying on $-a_{jk}$. Then we get

$$(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\cdot\prod_{k\neq i}\mathbf{P}_{i\,k}\left(-a_{j\,k}\right) = \begin{array}{c}\left(\begin{array}{ccccc} \sum\limits_{k\neq j} a^*_{1k}a_{k1} & \ldots & a^*_{1j} & \ldots & \sum\limits_{k\neq j} a^*_{1k}a_{kn} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum\limits_{k\neq j} a^*_{nk}a_{k1} & \ldots & a^*_{nj} & \ldots & \sum\limits_{k\neq j} a^*_{nk}a_{kn} \end{array}\right) \\ \scriptstyle i-th \end{array}.$$

The obtained matrix has the following factorization.

$$\begin{array}{c}\left(\begin{array}{ccccc} \sum\limits_{k\neq j} a^*_{1k}a_{k1} & \ldots & a^*_{1j} & \ldots & \sum\limits_{k\neq j} a^*_{1k}a_{kn} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum\limits_{k\neq j} a^*_{nk}a_{k1} & \ldots & a^*_{nj} & \ldots & \sum\limits_{k\neq j} a^*_{nk}a_{kn} \end{array}\right) \\ \scriptstyle i-th \end{array} =$$

$$= \begin{pmatrix} a^*_{11} & a^*_{12} & \ldots & a^*_{1m} \\ a^*_{21} & a^*_{22} & \ldots & a^*_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ a^*_{n1} & a^*_{n2} & \ldots & a^*_{nm} \end{pmatrix} \begin{array}{c}\left(\begin{array}{ccccc} a_{11} & \ldots & 0 & \ldots & a_{n1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & \ldots & 0 & \ldots & a_{mn} \end{array}\right) \\ \scriptstyle i-th \end{array} j-th.$$

Denote by $\tilde{\mathbf{A}} := \begin{array}{c}\left(\begin{array}{ccccc} a_{11} & \ldots & 0 & \ldots & a_{n1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & \ldots & 0 & \ldots & a_{mn} \end{array}\right) \\ \scriptstyle i-th \end{array} j-th$. The matrix $\tilde{\mathbf{A}}$ is obtained from $\mathbf{A}$ by replacing all entries of the jth row and of the ith column with zeroes except that the (j,i) entry equals 1. Elementary transformations of a matrix do not change its rank and the rank of a matrix product does not exceed a rank of each factors. It follows that $\operatorname{rank}\,(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right) \leq \min\left\{\operatorname{rank}\mathbf{A}^*, \operatorname{rank}\tilde{\mathbf{A}}\right\}$. It is obviously that $\operatorname{rank}\tilde{\mathbf{A}} \geq \operatorname{rank}\mathbf{A} = \operatorname{rank}\mathbf{A}^*$. Taking into account Theorem 9.7 we obtain $\operatorname{rank}\mathbf{A}^*\mathbf{A} = \operatorname{rank}\mathbf{A}$. This completes the proof. ■

The following lemma is proved in the same way.

Lemma 17.2. *If* $\mathbf{A} \in \mathbb{H}_r^{m\times n}$*, then* $\operatorname{rank}\left(\mathbf{A}\mathbf{A}^*\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right) \leq r$.

We shall use the following notations. Let $\alpha := \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\beta := \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\}$ be subsets of the order $1 \leq k \leq \min\{m, n\}$. By $\mathbf{A}^{\alpha}_{\beta}$ denote the submatrix of $\mathbf{A}$ determined by the rows indexed by α and the columns indexed by β. Then $\mathbf{A}^{\alpha}_{\alpha}$ denotes the principal submatrix determined by the rows and columns indexed by α. If $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is Hermitian, then by $|\mathbf{A}^{\alpha}_{\alpha}|$ denote the corresponding principal minor of $\det \mathbf{A}$. For $1 \leq k \leq n$, denote by $L_{k,n} := \{\,\alpha : \alpha = (\alpha_1, \ldots, \alpha_k)\,,\ 1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq n\}$ the collection of strictly increasing sequences of k integers chosen from $\{1, \ldots, n\}$. For fixed $i \in \alpha$ and $j \in \beta$, let $I_{r,\,m}\{i\} := \{\,\alpha : \alpha \in L_{r,m}, i \in \alpha\}$, $\quad J_{r,\,n}\{j\} := \{\,\beta : \beta \in L_{r,n}, j \in \beta\}$.

Lemma 17.3. *If* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $t \in \mathbb{R}$*, then*

$$\operatorname{cdet}_i\left(t\mathbf{I} + \mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{a}^*_{.j}\right) = c_1^{(ij)} t^{n-1} + c_2^{(ij)} t^{n-2} + \ldots + c_n^{(ij)}, \tag{44}$$

where $c_n^{(ij)} = \operatorname{cdet}_i\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right)$ *and* $c_k^{(ij)} = \sum\limits_{\beta\in J_{k,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right)\,{}^{\beta}_{\beta}$ *for all* $k = \overline{1, n-1}$*,* $i = \overline{1, n}$*, and* $j = \overline{1, m}$*.*

Proof. Denote by $\mathbf{b}_{.\,i}$ the ith column of the Hermitian matrix $\mathbf{A}^*\mathbf{A} =: \left(b_{ij}\right)_{n\times n}$. Consider the Hermitian matrix $\left(t\mathbf{I} + \mathbf{A}^*\mathbf{A}\right)_{.\,i}(\mathbf{b}_{.\,i}) \in \mathbb{H}^{n\times n}$. It differs from $\left(t\mathbf{I} + \mathbf{A}^*\mathbf{A}\right)$ an entry b_{ii}. Taking into account Theorem 16.1 we obtain

$$\det\left(t\mathbf{I} + \mathbf{A}^*\mathbf{A}\right)_{.i}(\mathbf{b}_{.\,i}) = d_1 t^{n-1} + d_2 t^{n-2} + \ldots + d_n, \tag{45}$$

where $d_k = \sum\limits_{\beta\in J_{k,\,n}\{i\}} \det\left(\mathbf{A}^*\mathbf{A}\right)\,{}^{\beta}_{\beta}$ is the sum of all principal minors of order k that contain the ith column for all $k = \overline{1, n-1}$ and $d_n = \det\left(\mathbf{A}^*\mathbf{A}\right)$. Consequently we have

$$\mathbf{b}_{.\,i} = \begin{pmatrix} \sum\limits_l a^*_{1l} a_{li} \\ \sum\limits_l a^*_{2l} a_{li} \\ \vdots \\ \sum\limits_l a^*_{nl} a_{li} \end{pmatrix} = \sum_l \mathbf{a}^*_{.l} a_{li},$$

where $\mathbf{a}^*_{.l}$ is the lth column-vector of $\mathbf{A}^*$ for all $l = \overline{1, m}$. Taking into account Theorem 4.1, Lemma 3.1 and Theorem 3.3 we obtain on the one hand

$$\begin{gathered} \det\left(t\mathbf{I} + \mathbf{A}^*\mathbf{A}\right)_{.i}(\mathbf{b}_{.\,i}) = \operatorname{cdet}_i\left(t\mathbf{I} + \mathbf{A}^*\mathbf{A}\right)_{.i}(\mathbf{b}_{.\,i}) = \\ = \sum_l \operatorname{cdet}_i\left(t\mathbf{I} + \mathbf{A}^*\mathbf{A}\right)_{.l}\left(\mathbf{a}^*_{.l} a_{li}\right) = \sum_l \operatorname{cdet}_i\left(t\mathbf{I} + \mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{a}^*_{.l}\right) \cdot a_{li} \end{gathered} \tag{46}$$

On the other hand having changed the order of summation, we get for all $k = \overline{1, n-1}$

$$\begin{gathered} d_k = \sum_{\beta\in J_{k,\,n}\{i\}} \det\left(\mathbf{A}^*\mathbf{A}\right)\,{}^{\beta}_{\beta} = \sum_{\beta\in J_{k,\,n}\{i\}} \operatorname{cdet}_i\left(\mathbf{A}^*\mathbf{A}\right)\,{}^{\beta}_{\beta} = \\ \sum_{\beta\in J_{k,\,n}\{i\}} \sum_l \operatorname{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.l} a_{li}\right)\right)^{\beta}_{\beta} = \sum_l \sum_{\beta\in J_{k,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.l}\right)\right)\,{}^{\beta}_{\beta} \cdot a_{li}. \end{gathered} \tag{47}$$

By substituting (46) and (47) in (45), and equating factors at $a_{l\,i}$ when $l = j$, we obtain the equality (44). ■

By analogy can be proved the following lemma.

Lemma 17.4. *If* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $t \in \mathbb{R}$*, then*

$$\mathrm{rdet}_j(t\mathbf{I} + \mathbf{A}\mathbf{A}^*)_{j\,.}(\mathbf{a}^*_{i.}) = r_1^{(ij)} t^{n-1} + r_2^{(ij)} t^{n-2} + \ldots + r_n^{(ij)},$$

where $r_n^{(ij)} = \mathrm{rdet}_j(\mathbf{A}\mathbf{A}^*)_{j\,.}(\mathbf{a}^*_{i.})$ *and* $r_k^{(ij)} = \sum\limits_{\alpha\in I_{r,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{A}\mathbf{A}^*)_{j\,.}(\mathbf{a}^*_{i.})\right)\,{}^{\alpha}_{\alpha}$ *for all* $k = \overline{1, n-1}$*,* $i = \overline{1,n}$*, and* $j = \overline{1,m}$*.*

Theorem 17.1. *If* $\mathbf{A} \in \mathbb{H}^{m\times n}_r$*, then the Moore-Penrose inverse* $\mathbf{A}^+ = \left(a^+_{ij}\right) \in \mathbb{H}^{n\times m}$ *possess the following determinantal representations:*

$$a^+_{ij} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right)\,{}^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\;n}} \left|\left(\mathbf{A}^*\mathbf{A}\right)\,{}^{\beta}_{\beta}\right|}, \tag{48}$$

or

$$a^+_{ij} = \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{A}\mathbf{A}^*)_{j\,.}(\mathbf{a}^*_{i.})\right)\,{}^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r,\;m}} \left|(\mathbf{A}\mathbf{A}^*)\,{}^{\alpha}_{\alpha}\right|}. \tag{49}$$

Proof. At first we prove (48). Using Theorem 16.3, we have

$$\mathbf{A}^+ = \lim_{\alpha\to 0}\left(\alpha\mathbf{I} + \mathbf{A}^*\mathbf{A}\right)^{-1}\mathbf{A}^*.$$

The matrix $(\alpha\mathbf{I} + \mathbf{A}^*\mathbf{A}) \in \mathbb{H}^{n\times n}$ is a full-rank Hermitian matrix. Taking into account Theorem 7.1 it has an inverse, which we represent as a left inverse matrix

$$\left(\alpha\mathbf{I} + \mathbf{A}^*\mathbf{A}\right)^{-1} = \frac{1}{\det\left(\alpha\mathbf{I} + \mathbf{A}^*\mathbf{A}\right)}\begin{pmatrix} L_{11} & L_{21} & \ldots & L_{n1} \\ L_{12} & L_{22} & \ldots & L_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ L_{1n} & L_{2n} & \ldots & L_{nn} \end{pmatrix},$$

where L_{ij} is a left ijth cofactor of a matrix $\alpha\mathbf{I} + \mathbf{A}^*\mathbf{A}$. Then we have

$$\left(\alpha\mathbf{I} + \mathbf{A}^*\mathbf{A}\right)^{-1}\mathbf{A}^* =$$

$$= \tfrac{1}{\det(\alpha\mathbf{I}+\mathbf{A}^*\mathbf{A})}\begin{pmatrix} \sum\limits_{k=1}^{n} L_{k1}a^*_{k1} & \sum\limits_{k=1}^{n} L_{k1}a^*_{k2} & \ldots & \sum\limits_{k=1}^{n} L_{k1}a^*_{km} \\ \sum\limits_{k=1}^{n} L_{k2}a^*_{k1} & \sum\limits_{k=1}^{n} L_{k2}a^*_{k2} & \ldots & \sum\limits_{k=1}^{n} L_{k2}a^*_{km} \\ \ldots & \ldots & \ldots & \ldots \\ \sum\limits_{k=1}^{n} L_{kn}a^*_{k1} & \sum\limits_{k=1}^{n} L_{kn}a^*_{k2} & \ldots & \sum\limits_{k=1}^{n} L_{kn}a^*_{km} \end{pmatrix}.$$

Using the definition of a left cofactor, we obtain

$$\mathbf{A}^{+} = \lim_{\alpha\to 0}\begin{pmatrix} \frac{\operatorname{cdet}_1(\alpha\mathbf{I}+\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.1})}{\det(\alpha\mathbf{I}+\mathbf{A}^*\mathbf{A})} & \dots & \frac{\operatorname{cdet}_1(\alpha\mathbf{I}+\mathbf{A}^*\mathbf{A})_{.1}(\mathbf{a}^*_{.m})}{\det(\alpha\mathbf{I}+\mathbf{A}^*\mathbf{A})} \\ \dots & \dots & \dots \\ \frac{\operatorname{cdet}_n(\alpha\mathbf{I}+\mathbf{A}^*\mathbf{A})_{.n}(\mathbf{a}^*_{.1})}{\det(\alpha\mathbf{I}+\mathbf{A}^*\mathbf{A})} & \dots & \frac{\operatorname{cdet}_n(\alpha\mathbf{I}+\mathbf{A}^*\mathbf{A})_{.n}(\mathbf{a}^*_{.m})}{\det(\alpha\mathbf{I}+\mathbf{A}^*\mathbf{A})} \end{pmatrix} \tag{50}$$

By Theorem 16.1 we have $\det(\alpha\mathbf{I}+\mathbf{A}^*\mathbf{A}) = \alpha^n + d_1\alpha^{n-1} + d_2\alpha^{n-2} + \ldots + d_n$, where $d_k = \sum\limits_{\beta\in J_{k,\,n}} \left|(\mathbf{A}^*\mathbf{A})\,{}^{\beta}_{\beta}\right|$ is a sum of principal minors of $\mathbf{A}^*\mathbf{A}$ of order k for all $k=\overline{1,n-1}$ and $d_n = \det\mathbf{A}^*\mathbf{A}$. Since $\operatorname{rank}\mathbf{A}^*\mathbf{A} = \operatorname{rank}\mathbf{A} = r$ and $d_n = d_{n-1} = \ldots = d_{r+1} = 0$, it follows that

$$\det(\alpha\mathbf{I}+\mathbf{A}^*\mathbf{A}) = \alpha^n + d_1\alpha^{n-1} + d_2\alpha^{n-2} + \ldots + d_r\alpha^{n-r}.$$

Using (44) we get

$$\operatorname{cdet}_i(\alpha\mathbf{I}+\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{a}^*_{.j}\right) = c_1^{(ij)}\alpha^{n-1} + c_2^{(ij)}\alpha^{n-2} + \ldots + c_n^{(ij)},$$

for all $i=\overline{1,n}$ and $j=\overline{1,m}$, where $c_k^{(ij)} = \sum\limits_{\beta\in J_{k,\,n}\{i\}} \operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right){}^{\beta}_{\beta}$ for all $k=\overline{1,n-1}$ and $c_n^{(ij)} = \operatorname{cdet}_i(\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{a}^*_{.j}\right)$.

Now we prove that $c_k^{(ij)} = 0$, when $k \geq r+1$ for all $i=\overline{1,n}$, and $j=\overline{1,m}$. By Lemma 17.1 $\operatorname{rank}(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right) \leq r$, then the matrix $(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)$ has no more r right-linearly independent columns.

Consider $\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right){}^{\beta}_{\beta}$, when $\beta\in J_{k,n}\{i\}$. It is a principal submatrix of $(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)$ of order $k\geq r+1$. Deleting both its ith row and column, we obtain a principal submatrix of order $k-1$ of $\mathbf{A}^*\mathbf{A}$. We denote it by $\mathbf{M}$. The following cases are possible.

1. If $k = r+1$ and $\det\mathbf{M}\neq 0$. In this case all columns of $\mathbf{M}$ are right-linearly independent. The addition of all of them on one coordinate to columns of $\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right){}^{\beta}_{\beta}$ keeps their right-linear independence. Hence, they are basis in a matrix $\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right){}^{\beta}_{\beta}$, and by Theorem 9.4 the ith column is the right linear combination of its basis columns. From this by Theorem 8.7, we get $\operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right){}^{\beta}_{\beta} = 0$, when $\beta\in J_{k,n}\{i\}$ and $k\geq r+1$.

2. If $k = r+1$ and $\det\mathbf{M} = 0$, than p, ($p<k$), columns are basis in $\mathbf{M}$ and in $\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right){}^{\beta}_{\beta}$. Then by Theorems 9.4 and 8.7 we obtain $\operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right){}^{\beta}_{\beta} = 0$ as well.

3. If $k > r+1$, then from Theorems 9.7 and 9.6 it follows that $\det\mathbf{M} = 0$ and p, ($p < k-1$), columns are basis in the both matrices $\mathbf{M}$ and $\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right){}^{\beta}_{\beta}$. Then by Theorems 9.4 and 8.7, we obtain that $\operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right){}^{\beta}_{\beta} = 0$.

Thus in all cases we have $\operatorname{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right)\,{}^{\beta}_{\beta}=0$, when $\beta\in J_{k,n}\{i\}$ and $r+1\leq k<n$. From here if $r+1\leq k<n$, then

$$c_k^{(ij)}=\sum_{\beta\in J_{k,\,n}\{i\}}\operatorname{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right)\,{}^{\beta}_{\beta}=0,$$

and

$$c_n^{(ij)}=\operatorname{cdet}_i\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right)=0$$

for all $i=\overline{1,n}$ and $j=\overline{1,m}$. Hence,

$$\operatorname{cdet}_i\left(\alpha\mathbf{I}+\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right)=c_1^{(ij)}\alpha^{n-1}+c_2^{(ij)}\alpha^{n-2}+\ldots+c_r^{(ij)}\alpha^{n-r}$$

for all $i=\overline{1,n}$ and $j=\overline{1,m}$. By substituting these values in the matrix from (50), we obtain

$$\mathbf{A}^+=\lim_{\alpha\to 0}\begin{pmatrix}\frac{c_1^{(11)}\alpha^{n-1}+\ldots+c_r^{(11)}\alpha^{n-r}}{\alpha^n+d_1\alpha^{n-1}+\ldots+d_r\alpha^{n-r}} & \ldots & \frac{c_1^{(1m)}\alpha^{n-1}+\ldots+c_r^{(1m)}\alpha^{n-r}}{\alpha^n+d_1\alpha^{n-1}+\ldots+d_r\alpha^{n-r}}\\ \ldots & \ldots & \ldots\\ \frac{c_1^{(n1)}\alpha^{n-1}+\ldots+c_r^{(n1)}\alpha^{n-r}}{\alpha^n+d_1\alpha^{n-1}+\ldots+d_r\alpha^{n-r}} & \ldots & \frac{c_1^{(nm)}\alpha^{n-1}+\ldots+c_r^{(nm)}\alpha^{n-r}}{\alpha^n+d_1\alpha^{n-1}+\ldots+d_r\alpha^{n-r}}\end{pmatrix}=$$
$$\begin{pmatrix}\frac{c_r^{(11)}}{d_r} & \ldots & \frac{c_r^{(1m)}}{d_r}\\ \ldots & \ldots & \ldots\\ \frac{c_r^{(n1)}}{d_r} & \ldots & \frac{c_r^{(nm)}}{d_r}\end{pmatrix}.$$

Here $c_r^{(ij)}=\sum\limits_{\beta\in J_{r,\,n}\{i\}}\operatorname{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right)\,{}^{\beta}_{\beta}$ and $d_r=\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^*\mathbf{A}\right)\,{}^{\beta}_{\beta}\right|$. Thus, we have obtained the determinantal representation of $\mathbf{A}^+$ by (48).

By analogy can be proved the determinantal representation of $\mathbf{A}^+$ by (49). ■

Remark 17.1. *In (48) the index i in $\operatorname{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right)\,{}^{\beta}_{\beta}$ designates ith column of $\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right)$, but in the submatrix $\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right)\,{}^{\beta}_{\beta}$ the entries of $\mathbf{a}^*_{.j}$ may be placed in a column with the another index. In (49) we have equivalently.*

Remark 17.2. *If $\operatorname{rank}\mathbf{A}=n$, then by Corollary 16.1 $\mathbf{A}^+=\left(\mathbf{A}^*\mathbf{A}\right)^{-1}\mathbf{A}^*$. Considering $\left(\mathbf{A}^*\mathbf{A}\right)^{-1}$ as a left inverse, we get the following representation of $\mathbf{A}^+$:*

$$\mathbf{A}^+=\frac{1}{\operatorname{ddet}\mathbf{A}}\begin{pmatrix}\operatorname{cdet}_1(\mathbf{A}^*\mathbf{A})_{.\,1}\left(\mathbf{a}^*_{.1}\right) & \ldots & \operatorname{cdet}_1(\mathbf{A}^*\mathbf{A})_{.\,1}\left(\mathbf{a}^*_{.\,m}\right)\\ \ldots & \ldots & \ldots\\ \operatorname{cdet}_n(\mathbf{A}^*\mathbf{A})_{.\,n}\left(\mathbf{a}^*_{.1}\right) & \ldots & \operatorname{cdet}_n(\mathbf{A}^*\mathbf{A})_{.\,n}\left(\mathbf{a}^*_{.\,m}\right).\end{pmatrix}\tag{51}$$

If $m>n$, then by Theorem 17.1 for $\mathbf{A}^+$ we have (49) as well.

Remark 17.3. *If $\operatorname{rank}\mathbf{A}=m$, then by Corollary 16.1 $\mathbf{A}^+=\mathbf{A}^*\left(\mathbf{A}\mathbf{A}^*\right)^{-1}$. Considering $\left(\mathbf{A}\mathbf{A}^*\right)^{-1}$ as a right inverse, we get the following representation of $\mathbf{A}^+$:*

$$\mathbf{A}^+=\frac{1}{\operatorname{ddet}\mathbf{A}}\begin{pmatrix}\operatorname{rdet}_1(\mathbf{A}\mathbf{A}^*)_{1.}\left(\mathbf{a}^*_{1.}\right) & \ldots & \operatorname{rdet}_m(\mathbf{A}\mathbf{A}^*)_{m.}\left(\mathbf{a}^*_{1.}\right)\\ \ldots & \ldots & \ldots\\ \operatorname{rdet}_1(\mathbf{A}\mathbf{A}^*)_{1.}\left(\mathbf{a}^*_{n.}\right) & \ldots & \operatorname{rdet}_m(\mathbf{A}\mathbf{A}^*)_{m\,.}\left(\mathbf{a}^*_{n\,.}\right)\end{pmatrix}.\tag{52}$$

If $m<n$, then by Theorem 17.1 for $\mathbf{A}^+$ we also have (48).

Corollary 17.1. *If* $\mathbf{A} \in \mathbb{H}_r^{m\times n}$*, where* $r < \min\{m,n\}$ *or* $r = m < n$*, then for a projection matrix* $\mathbf{A}^+\mathbf{A} =: \mathbf{P} = (p_{ij})_{n\times n}$ *we have its following determinantal representation*

$$p_{ij} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{d}_{.j}\right)\right){}^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|(\mathbf{A}^*\mathbf{A})\,{}^{\beta}_{\beta}\right|},$$

where $\mathbf{d}_{.j}$ *is the* j*-th column of* $\mathbf{A}^*\mathbf{A} \in \mathbb{H}^{n\times n}$ *and for all* $i,j = \overline{1,n}$*.*

Proof. Representing $\mathbf{A}^+$ by (48) and right-multiplying it by $\mathbf{A}$, we obtain for an entry p_{ij} of $\mathbf{A}^+\mathbf{A} =: \mathbf{P} = (p_{ij})_{n\times n}$.

$$p_{ij} = \sum_{p=1}^{m} c_{ip}\cdot a_{pj} = \sum_{k} \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right){}^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|(\mathbf{A}^*\mathbf{A})\,{}^{\beta}_{\beta}\right|}\cdot a_{kj}$$

$$= \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}}\sum\limits_{k} \operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right){}^{\beta}_{\beta}\cdot a_{kj}}{\sum\limits_{\beta\in J_{r,\,n}} \left|(\mathbf{A}^*\mathbf{A})\,{}^{\beta}_{\beta}\right|} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{d}_{.j}\right)\right){}^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|(\mathbf{A}^*\mathbf{A})\,{}^{\beta}_{\beta}\right|},$$

where $d_{.j}$ is the j-th column of $\mathbf{A}^*\mathbf{A} \in \mathbb{H}^{n\times n}$ and for all $i,j = \overline{1,n}$. ■

By analogy can be proved the following corollary.

Corollary 17.2. *If* $\mathbf{A} \in \mathbb{H}_r^{m\times n}$*, where* $r < \min\{m,n\}$ *or* $r = n < m$*, then for the projection matrix* $\mathbf{A}\mathbf{A}^+ =: \mathbf{Q} = (q_{ij})_{m\times m}$ *we have its following determinantal representation*

$$q_{ij} = \frac{\sum\limits_{\alpha\in I_{r,\,m}\{i\}} \left|\left((\mathbf{A}\,\mathbf{A}^*)_{i.}\left(\mathbf{g}_{j.}\right)\right){}^{\alpha}_{\alpha}\right|}{\sum\limits_{\alpha\in I_{r,\,m}} \left|(\mathbf{A}\mathbf{A}^*)\,{}^{\alpha}_{\alpha}\right|},$$

where $\mathbf{g}_{j.}$ *is the* j*-th row of* $(\mathbf{A}\mathbf{A}^*) \in \mathbb{H}^{m\times m}$ *and for all* $i,j = \overline{1,m}$*.*

Remark 17.4. *By definition of a classical adjoint matrix of* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *can be put* $\operatorname{Adj}[\mathbf{A}]\cdot\mathbf{A} = \mathbf{A}\cdot\operatorname{Adj}[\mathbf{A}] = \det\mathbf{A}\cdot\mathbf{I}$*. Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$*. If* $\operatorname{rank}\mathbf{A} = n$*, the by Corollary 16.1 we have* $\mathbf{A}^+\mathbf{A} = \mathbf{I}_n$*. Representing* $\mathbf{A}^+$ *by (51) as* $\mathbf{A}^+ = \frac{\mathbf{L}}{\det(\mathbf{A}^*\mathbf{A})}$*, where* $\mathbf{L} = \left(\operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right)\right)_{n\times m}$*, we obtain* $\mathbf{L}\mathbf{A} = \det(\mathbf{A}^*\mathbf{A})\cdot\mathbf{I}_n$*. This means that the matrix* $\mathbf{L} =: \operatorname{Adj}_L[\mathbf{A}]$ *is the left classical adjoint matrix of* $\mathbf{A} \in \mathbb{H}^{m\times n}$*, i.e.*

$$\operatorname{Adj}_L[\mathbf{A}] = \left(\operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right)\right)_{n\times m}.$$

If $\operatorname{rank}\mathbf{A} = m$*, then by definition of a right classical adjoint matrix of* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *by Corollary 16.1 and by (52) we can put*

$$\operatorname{Adj}_R[\mathbf{A}] := \left(\left(\operatorname{rdet}_j(\mathbf{A}\mathbf{A}^*)_{j.}(\mathbf{a}^*_{i.})\right)\right)_{m\times n}.$$

Since in this case $\mathbf{A} \cdot \mathrm{Adj}\,_R[\mathbf{A}] = \det(\mathbf{A}\mathbf{A}^*) \cdot \mathbf{I}$.

If $\operatorname{rank}\mathbf{A} = r < \min\{m, n\}$, *then an analog of a left classical adjoint matrix of* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *by (48) can accept*

$$\mathrm{Adj}\,_L\,[\mathbf{A}] := \left(\sum_{\alpha\in J_{r,\,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.\,i}\,\mathbf{a}^*_{.j} \right)\,{}^{\alpha}_{\alpha} \right)_{n\times m}.$$

Indeed, since eigenvalues of a projection matrix are only 1 and 0, then there exists such an unitary matrix $\mathbf{U} \in \mathbb{H}^{n\times n}$ *that*

$$\begin{gathered} \mathrm{Adj}\,_L[\mathbf{A}] \cdot \mathbf{A} = \sum_{\alpha\in I_{r,\,n}} |(\mathbf{A}^*\mathbf{A})\,{}^{\alpha}_{\alpha}| \cdot \mathbf{P} = \\ = \sum_{\alpha\in I_{r,\,n}} |(\mathbf{A}^*\mathbf{A})\,{}^{\alpha}_{\alpha}| \cdot \mathbf{U}\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)\mathbf{U}^*. \end{gathered}$$

If $\operatorname{rank}\mathbf{A} = r < \min\{m, n\}$, *then by an analogue of a right classical adjoint matrix of* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *by (49) we can put*

$$\mathrm{Adj}\,_R\,[\mathbf{A}] := \left(\sum_{\alpha\in I_{r,\,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{A}\mathbf{A}^*)_{j.}\,(\mathbf{a}^*_{i.})\,\right)^{\alpha}_{\alpha} \right)_{n\times m}.$$

Indeed, then there exists such unitary matrix $\mathbf{V} \in \mathbb{H}^{m\times m}$ *that*

$$\begin{gathered} \mathbf{A} \cdot \mathrm{Adj}\,_R\,[\mathbf{A}] = \sum_{\alpha\in J_{r,\,m}} |(\mathbf{A}\mathbf{A}^*)\,{}^{\alpha}_{\alpha}| \cdot \mathbf{Q} = \\ = \sum_{\alpha\in J_{r,\,m}} |(\mathbf{A}\mathbf{A}^*)\,{}^{\alpha}_{\alpha}| \cdot \mathbf{V}\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)\mathbf{V}^*. \end{gathered}$$

Remark 17.5. *If* $\mathbf{A} \in \mathbb{C}^{m\times n}$ *is a matrix with complex entries, then we obtain the following analogs of (48) and (49), respectively,*

$$a^+_{ij} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \left| \left((\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\right)\,{}^{\beta}_{\beta} \right|}{\sum\limits_{\beta\in J_{r,\,n}} \left| (\mathbf{A}^*\mathbf{A})\,{}^{\beta}_{\beta} \right|}, \quad a^+_{ij} = \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \left| \left((\mathbf{A}\mathbf{A}^*)_{j\,.}\,(\mathbf{a}^*_{i.}) \right)\,{}^{\alpha}_{\alpha} \right|}{\sum\limits_{\alpha\in I_{r,\,m}} |(\mathbf{A}\mathbf{A}^*)\,{}^{\alpha}_{\alpha}|}$$

for all $i = \overline{1, n}$ *and* $j = \overline{1, m}$. *These determinantal representations are original in this case as well. It is reflected in [20].*

18. Cramer's Rule for a Least Squares Solution of Quaternion System Linear Equations

Definition 18.1. *Suppose*

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{y}, \tag{53}$$

is a right system linear equations over the quaternion skew field $\mathbb{H}$, *where* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *is the coefficient matrix,* $\mathbf{y} \in \mathbb{H}^{m\times 1}$ *is a column of constants, and* $\mathbf{x} \in \mathbb{H}^{n\times 1}$ *is a column of* *unknown. The least squares solution of (53) (with the least norm) is called the vector* $\mathbf{x}^0$ *satisfying*

$$\| \mathbf{x}^0 \| = \min_{\mathbf{x}\in\mathbb{H}^n} \left\{ \| \tilde{\mathbf{x}} \| : \| \mathbf{A}\cdot\tilde{\mathbf{x}} - \mathbf{y} \| = \min_{\mathbf{x}\in\mathbb{H}^n} \| \mathbf{A}\cdot\mathbf{x} - \mathbf{y} \| \right\},$$

where $\mathbb{H}^n$ *is an* n*-dimension right quaternion vector space.*

We recall that in the right quaternion vector space $\mathbb{H}^n$ by definition of the inner product of vectors we put $\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{y}^*\mathbf{x} = \overline{y_1}\cdot x_1 \cdot + \ldots + \overline{y_n}\cdot x_n$ and $\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ is the norm of a vector $\mathbf{x} \in \mathbb{H}^n$. By analogy to a complex case (see, e.g. [13]) we can prove the following theorem.

Theorem 18.1. *The vector* $\mathbf{x} = \mathbf{A}^+\mathbf{y}$ *is the least square solution of (53).*

Definition 18.2. *Suppose*

$$\mathbf{x}\cdot\mathbf{A} = \mathbf{y}, \tag{54}$$

is a left system linear equations over the quaternion skew field $\mathbb{H}$*, where* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *is the coefficient matrix,* $\mathbf{y} \in \mathbb{H}^{1\times n}$ *is a row of constants, and* $\mathbf{x} \in \mathbb{H}^{1\times m}$ *is a row of unknown. The least squares solution of (54) (with the least norm) is called the vector* $\mathbf{x}^0$ *satisfying*

$$\left\|\mathbf{x}^0\right\| = \min_{\tilde{\mathbf{x}}\in {}^m\mathbb{H}} \left\{ \|\tilde{\mathbf{x}}\| : \ \|\tilde{\mathbf{x}}\cdot\mathbf{A} - \mathbf{y}\| = \min_{\mathbf{x}\in {}^m\mathbb{H}} \|\mathbf{x}\cdot\mathbf{A} - \mathbf{y}\| \right\},$$

where ${}^m\mathbb{H}$ *is an* m*-dimension left quaternion vector space.*

We recall that in the left quaternion vector space ${}^m\mathbb{H}$ by definition of the inner product of vectors we can put $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}\mathbf{y}^* = x_1\cdot\overline{y_1} + \ldots + x_m\cdot\overline{y_m}$. Then $\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ is the norm of $\mathbf{x} \in {}^m\mathbb{H}$.

Theorem 18.2. *The vector* $\mathbf{x} = \mathbf{y}\cdot\mathbf{A}^+$ *is the least square solution of (54).*

Theorem 18.3. *(i) If* $\operatorname{rank}\mathbf{A} = n$*, then for the least square solution* $\mathbf{x}^0 = (x_1^0, \ldots, x_n^0)^T$ *of (53) we get for all* $j = \overline{1, n}$

$$x_j^0 = \frac{\operatorname{cdet}_j (\mathbf{A}^*\mathbf{A})_{.j}(\mathbf{f})}{\operatorname{ddet}\mathbf{A}}, \tag{55}$$

where $\mathbf{f} = \mathbf{A}^*\mathbf{y}$.

(ii) If $\operatorname{rank}\mathbf{A} = k \leq m < n$*, then for all* $j = \overline{1, n}$ *we have*

$$x_j^0 = \frac{\sum\limits_{\beta\in J_{r,\,n}\{j\}} \operatorname{cdet}_j \left((\mathbf{A}^*\mathbf{A})_{.j}(\mathbf{f}) \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left| (\mathbf{A}^*\mathbf{A}) {}_{\beta}^{\beta} \right|}. \tag{56}$$

Proof. i) If $\operatorname{rank}\mathbf{A} = n$, then $\mathbf{A}^+$ can be represented by (51). Denote $\mathbf{f} := \mathbf{A}^*\mathbf{y}$. Representing $\mathbf{A}^+\mathbf{y}$ by coordinates we obtain (55).
ii) If $\operatorname{rank}\mathbf{A} = k \leq m < n$, then by Theorem 17.1 we represent the matrix $\mathbf{A}^+$ by (48). Representing $\mathbf{A}^+\mathbf{y}$ by coordinates we obtain (56). ■

Remark 18.1. *In a complex case the following analogs of (55) and (56) are obtained respectively in [20] for all* $j = \overline{1, n}$,

$$x_j^0 = \frac{\det\left(\mathbf{A}^*\mathbf{A}\right)_{.j}(\mathbf{f})}{\det\left(\mathbf{A}^*\mathbf{A}\right)},\; x_j^0 = \frac{\sum\limits_{\beta \in J_{r,\,n}\{j\}} \left|\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.\,j}(\mathbf{f})\right)\,{}^{\beta}_{\beta}\right|}{\sum\limits_{\beta \in J_{r,\,n}} \left|\left(\mathbf{A}^*\mathbf{A}\right)\,{}^{\beta}_{\beta}\right|}.$$

Theorem 18.4. *(i) If* $\operatorname{rank}\mathbf{A} = m$, *then for* $\mathbf{x}^0 = (x_1^0, \ldots, x_m^0)$ *of (54) we obtain for all* $i = \overline{1, m}$

$$x_i^0 = \frac{\operatorname{rdet}_i\left(\mathbf{A}\mathbf{A}^*\right)_{i.}(\mathbf{z})}{\operatorname{ddet}\mathbf{A}}, \tag{57}$$

where $\mathbf{z} = \mathbf{y}\mathbf{A}^*$.

(ii) If $\operatorname{rank}\mathbf{A} = k \leq n < m$, *then for all* $i = \overline{1, m}$ *we have*

$$x_i^0 = \frac{\sum\limits_{\alpha \in I_{r,m}\{i\}} \operatorname{rdet}_i\left(\left(\mathbf{A}\mathbf{A}^*\right)_{i\,.}(\mathbf{z})\right)\,{}^{\alpha}_{\alpha}}{\sum\limits_{\alpha \in I_{r,\,m}} \left|\left(\mathbf{A}\mathbf{A}^*\right)\,{}^{\alpha}_{\alpha}\right|}. \tag{58}$$

The proof of this theorem is analogous to that of Theorem 18.3.

Remark 18.2. *In a complex case the following analogs of (57) and (58) respectively are obtained in [20] for all* $i = \overline{1, m}$,

$$x_i^0 = \frac{\det\left(\mathbf{A}\mathbf{A}^*\right)\,_{i.}(\mathbf{z})}{\det\mathbf{A}\mathbf{A}^*},\; x_i^0 = \frac{\sum\limits_{\alpha \in I_{r,m}\{i\}} \left|\left(\left(\mathbf{A}\mathbf{A}^*\right)_{i\,.}(\mathbf{z})\right)\,{}^{\alpha}_{\alpha}\right|}{\sum\limits_{\alpha \in I_{r,\,m}} \left|\left(\mathbf{A}\mathbf{A}^*\right)\,{}^{\alpha}_{\alpha}\right|}.$$

19. Example 2

Let us consider the left system of linear equations.

$$\begin{cases} x_1 i + 2x_2 i - x_3 = i, \\ -x_1 k + x_2 j + x_3 j = j, \\ x_1 j + x_2 + x_3 k = k, \\ x_1 + x_2 k + x_3 i = 1. \end{cases} \tag{59}$$

The coefficient matrix of the system is the matrix $\mathbf{A} = \begin{pmatrix} i & -k & j & 1 \\ 2i & j & 1 & k \\ -1 & j & k & i \end{pmatrix}$. The row of unknown is $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ and the row of constants is $\mathbf{y} = \begin{pmatrix} i & j & k & 1 \end{pmatrix}$. Then for (59) we have $\mathbf{x} \cdot \mathbf{A} = \mathbf{y}$. We obtain

$$\mathbf{A}^* = \begin{pmatrix} -i & -2i & -1 \\ k & -j & -j \\ -j & 1 & -k \\ 1 & -k & -i \end{pmatrix},$$

$$\mathbf{AA}^* = \begin{pmatrix} 4 & 2-i+j-k & -4i \\ 2+i-j+k & 7 & 1-2i-j-k \\ 4i & 1+2i+j+k & 4 \end{pmatrix}.$$

Since $\mathrm{ddet}\mathbf{A} = \det \mathbf{AA}^* = \mathrm{rdet}_1 \mathbf{AA}^* = 0$ and

$$\det\left(\mathbf{AA}^*\right)^{33} = \mathrm{rdet}_1 \begin{pmatrix} 4 & 2-i+j-k \\ 2+i-j+k & 7 \end{pmatrix} =$$
$$= 4\cdot 7 - (2-i+j-k)\cdot(2+i-j+k) = 21 \neq 0,$$

then by Theorem 9.7 $\mathrm{rank}\,\mathbf{A} = 2$. We shall represent $\mathbf{A}^+$ by (49).

$$\sum_{\alpha\in I_{2,\,3}} \left|\left(\mathbf{AA}^*\right)^{\alpha}_{\alpha}\right| = \det\begin{pmatrix} 4 & 2-i+j-k \\ 2+i-j+k & 7 \end{pmatrix} +$$
$$+\det\begin{pmatrix} 7 & 1-2i-j-k \\ 1+2i+j+k & 4 \end{pmatrix} + \det\begin{pmatrix} 4 & -4i \\ 4i & 4 \end{pmatrix} = 42.$$

Now we shall calculate $r_{ij} = \sum\limits_{\alpha\in I_{2,3}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{AA}^*\right)_{j\,.}(\mathbf{a}^*_{i.})\right)^{\alpha}_{\alpha}$ for all $i=\overline{1,4}$ and $j=\overline{1,3}$. To obtain r_{11}, we consider the matrix

$$\left(\mathbf{AA}^*\right)_{1\,.}(\mathbf{a}^*_{1.}) = \begin{pmatrix} -i & -2i & -1 \\ 2+i-j+k & 7 & 1-2i-j-k \\ 4i & 1+2i+j+k & 4 \end{pmatrix}.$$

Then we have

$$r_{11} = \mathrm{rdet}_1\begin{pmatrix} -i & -2i \\ 2+i-j+k & 7 \end{pmatrix} + \mathrm{rdet}_1\begin{pmatrix} -i & -1 \\ 4i & 4 \end{pmatrix} =$$
$$= -i\cdot 7 - (-2i)\cdot(2+i-j+k) - i\cdot 4 - (-1\cdot 4i) = -2-3i-2j-2k,$$

and so forth. Continuing in the same way, we get

$$\mathbf{A}^+ = \frac{1}{42}\begin{pmatrix} -2-3i-2j-2k & 2-12i+2j+2k & -3+2i+2j-2k \\ 1+i+2j+6k & -2+2i-6j-4k & 1-i-6j+2k \\ -2-i-6j-k & 6-2i+4j+2k & -1+2i+j-6k \\ 6+i+j+2k & -4+2i-2j-6k & 1-6i-2j+k \end{pmatrix}.$$

We find the least square solution by means of the matrix method by Theorem 18.2,

$$\mathbf{x}^0 = \mathbf{y}\cdot\mathbf{A}^+ = \frac{1}{42}\begin{pmatrix} 8+11i+3j-3k, & 12-4i-8j, & 11-8i+3j+3k \end{pmatrix}.$$

Now we shall find the least square solution of (59) by means of Cramer's rule by (58). We have $\mathbf{z} = \mathbf{y}\cdot\mathbf{A}^* = \begin{pmatrix} 2+2i, & 3, & 2-2i \end{pmatrix}$. Since

$$\left(\mathbf{AA}^*\right)_{1\,.}(\mathbf{z}) = \begin{pmatrix} 2+2i & 3 & 2-2i \\ 2+i-j+k & 7 & 1-2i-j-k \\ 4i & 1+2i+j+k & 4 \end{pmatrix},$$

then

$$x_1^0 = \frac{1}{\sum\limits_{\alpha\in I_{2,\,3}} \left|(\mathbf{AA}^*)\,{}_{\alpha}^{\alpha}\right|}\left(\mathrm{rdet}_1\begin{pmatrix}2+2i & 3\\ 2+i-j+k & 7\end{pmatrix} + \mathrm{rdet}_1\begin{pmatrix}2+2i & 2-2i\\ 4i & 4\end{pmatrix}\right)$$

$$= \frac{8+11i+3j-3k}{42}.$$

Since

$$(\mathbf{AA}^*)_{2\,.}(\mathbf{z}) = \begin{pmatrix}4 & 2-i+j-k & -4i\\ 2+2i & 3 & 2-2i\\ 4i & 1+2i+j+k & 4\end{pmatrix},$$

then

$$x_2^0 = \frac{1}{\sum\limits_{\alpha\in I_{2,\,3}} \left|(\mathbf{AA}^*)\,{}_{\alpha}^{\alpha}\right|}\left(\mathrm{rdet}_2\begin{pmatrix}4 & 2-i+j-k\\ 2+2i & 3\end{pmatrix}\right.$$

$$\left. + \mathrm{rdet}_1\begin{pmatrix}3 & 2-2i\\ 1+2i+j+k & 4\end{pmatrix}\right) = \frac{12-4i-8j}{42}.$$

Since

$$(\mathbf{AA}^*)_{3\,.}(\mathbf{z}) = \begin{pmatrix}4 & 2-i+j-k & -4i\\ 2+i-j+k & 7 & 1-2i-j-k\\ 2+2i & 3 & 2-2i\end{pmatrix},$$

then

$$x_3^0 = \frac{1}{\sum\limits_{\alpha\in I_{2,\,3}} \left|(\mathbf{AA}^*)\,{}_{\alpha}^{\alpha}\right|}\left(\mathrm{rdet}_2\begin{pmatrix}4 & -4i\\ 2+2i & 2-2i\end{pmatrix} + \mathrm{rdet}_1\begin{pmatrix}7 & 1-2i-j-k\\ 3 & 2-2i\end{pmatrix}\right)$$

$$= \frac{11-8i+3j+3k}{42}.$$

As you would expect, the solutions of (59) by matrix method and Cramer's rule coincided.

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