# Determinantal Representations of the Drazin and W-Weighted Drazin Inverses Over the Quaternion Skew Field...

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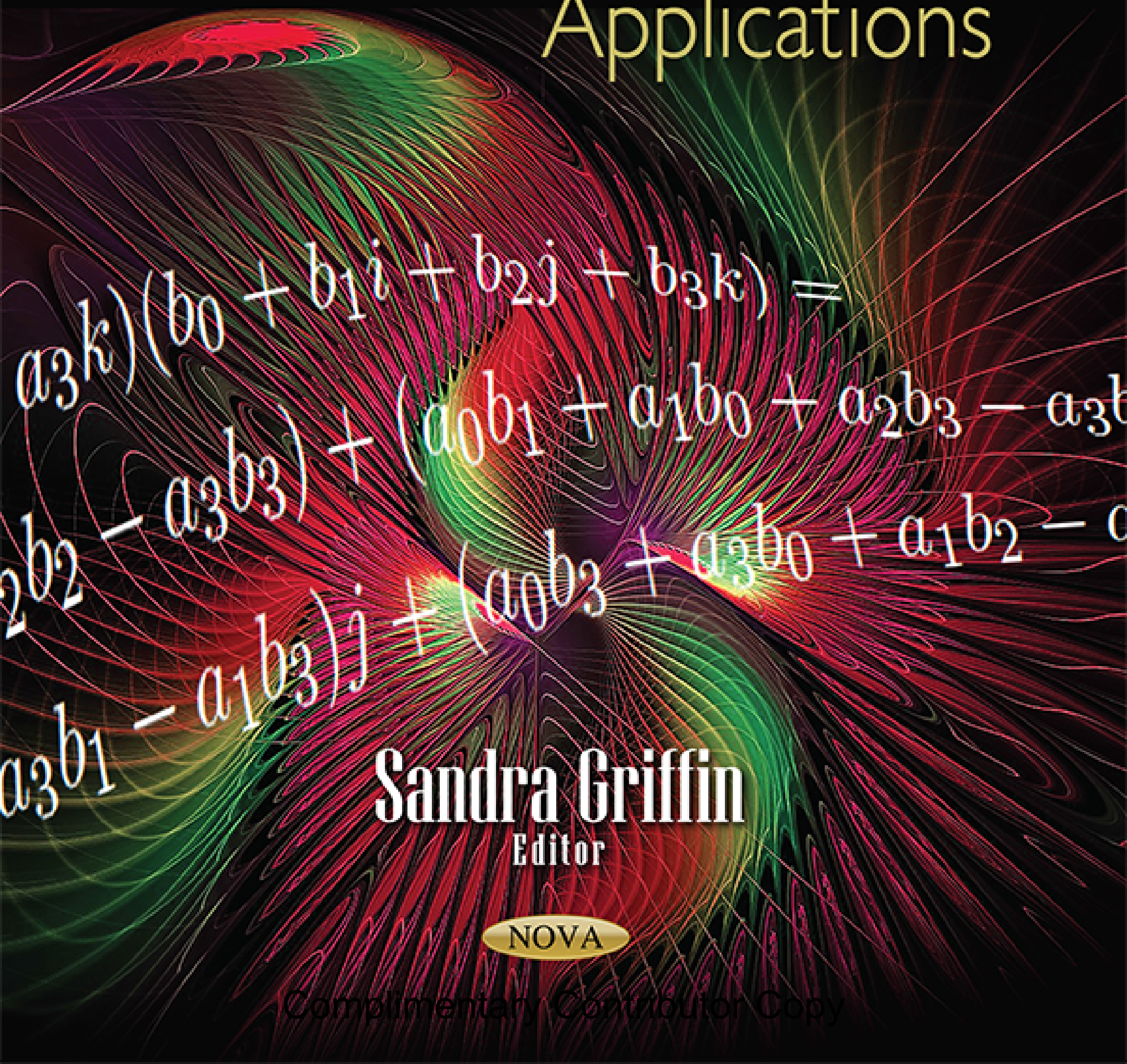
# Quaternions

## Theory and Applications

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# QUATERNIONS

## THEORY AND APPLICATIONS

SANDRA GRIFFIN
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# CONTENTS

# PREFACE

This book focuses on the theory and applications of quaternions. Chapter One collects some old problems on lattice orders and directed partial orders on complex numbers and quaternions, and summarizes recent development in answering those questions. Chapter Two discusses spin 1 particles with anomalous magnetic moments in the external uniform electric field. Chapter Three examines techniques of projective operators used to construct solutions for a spin 1 particle with anomalous magnetic moment in the external uniform magnetic field. Chapter Four analyzes the implementation of a cheap Micro AHRS (Attitude and Heading Reference System) using low-cost inertial sensors. Chapter Five reviews the basic concepts of quaternion and reduced biquaternions algebra. It introduces the 2D Hermite-Gaussian functions (2D-HGF) as the eigenfunction of discrete quaternion Fourier transform (DQFT) and discrete reduced biquaternion Fourier transform (DRBQFT), and the eigenvalues of two dimensional Hermite-Gaussian functions for three types of DQFT and two types of DRBQFT. Chapter Six investigates a leader-follower formation control problem of quadrotors. Chapter Seven considers determinantal representations the Drazin and weighted Drazin inverses over the quaternion skew field.

Chapter 1 collects some old problems on lattice orders and directed partial orders on complex numbers and quaternions, and summarizes recent development in answering those questions. Within the matrix 10-dimensional Duffin-Kemmer-Petiau formalism applied to the Shamaly-Capri field, Chapter 2 studies the behavior of a vector particle with anomalous magnetic moment in the presence of an external uniform electric field. The separation of variables in the wave equation is performed by using projective operator techniques and the theory of DKP-algebras. The whole wave function is decomposed into the sum of three components $\Psi_0$, $\Psi_+$, $\Psi_+$. It is enough to solve the equation for the main component $\Phi_0$, the two remaining ones being uniquely determined by it. Consequently, the problem reduces to three independent differential equations for three functions, which are of the type of one-dimensional Klein-Fock-Gordon equation in the presence of a uniform electric field modified by the non-vanishing anomalous magnetic moment of the particle. The solutions are constructed in terms of confluent hypergeometric functions. For assigning physical sense for these solutions, one must impose special restrictions on a certain parameter related to the anomalous moment of the particle. The neutral spin 1 particle is considered as well. In this case, the main manifestation of the anomalous magnetic moment consists in the modification of the ordinary plane wave solution along the electric field direction. Again, one must impose special restrictions on a parameter related to the anomalous moment of the particle.

Chapter 2 - Within *the matrix 10-dimensional Duffin-Kemmer-Petiau formalism* applied to *the Shamaly-Capri field*, Chapter 3 studies the behavior of a vector particle with anomalous magnetic moment in presence of an *external uniform magnetic field*. The separation of variables in the wave equation is performed by using projective operator techniques and the theory of DKP-algebras. The whole wave function is decomposed into the sum of three components $\Psi_0, \Psi_{+}, \Psi_{+}$. It is enough to solve the equation for the main component $\Phi_0$, the two remaining ones being uniquely determined by it. Consequently, the problem reduces to three independent differential equations for three functions, which are of the type of one-dimensional Klein--Fock--Gordon equation in the presence of a uniform electric field modified by the non-vanishing anomalous magnetic moment of the particle. The solutions are constructed in terms of confluent hypergeometric functions. For assigning physical sense for these solutions, one must impose special restrictions on a certain parameter related to the anomalous moment of the particle. The neutral spin 1 particle is considered as well. In this case, the main manifestation of the anomalous magnetic moment consists in the modification of the ordinary plane wave solution along the electric field direction. Again, one must impose special restrictions on a parameter related to the anomalous moment of the particle.

Chapter 3 - The separation of variables in the wave equation is performed using projective operator techniques and the theory of DKP-algebras. The problem is reduced to a system of 2-nd order differential equations for three independent functions, which is solved in terms of confluent hypergeometric functions. Three series of energy levels are found, of which two substantially differ from those for spin 1 particles without anomalous magnetic moment. For assigning to them physical sense for all the values of the main quantum number $n = 0, 1, 2, \ldots$, one must impose special restrictions on a parameter related to the anomalous moment. Otherwise, only some part of the energy levels corresponds to bound states. The neutral spin 1 particle is considered as well. In this case no bound states exist in the system, and the main qualitative manifestation of the anomalous magnetic moment consists in the occurrence of a space scaling of the arguments of the wave functions, compared to a particle without such a moment. Traditionally, the automotive industry has been the largest employer of robots, but their control is inline and programmed to follow planning trajectories. As shown in Chapter 4, in this case, in the department motor's test of Volkswagen Mexico a semi-autonomous robot is developed. To date, some critical technical problems must be solved in a number of areas, including in dynamics control. Generally, the attitude estimation and the measurement of the angular velocity are a requirements for the attitude control. As a result, the computational cost and the complexity of the control loop is relatively high.

Chapter 4 deals with the implementation of a cheap Micro AHRS (Attitude and Heading Reference System) using low-cost inertial sensors. In Chapter 4, the technique proposed is designed with attitude estimation and the prediction movement via the kinematic of a 4GDL robot. With this approach, only the measurements of at least two non-collinear directional sensors are needed. Since the control laws are highly simple and a model-based observer for angular velocity reconstruction is not needed, the proposed new strategy is very suitable for embedded implementations. The global convergence of the estimation and prediction techniques is proved. Simulation with some robustness tests is performed.

Chapter 5 - The quaternions, reduced biquaternions (RBs) and their respective Fourier transformations, i.e., discrete quaternion Fourier transform (DQFT) and discrete reduced biquaternion Fourier transform (DRBQFT), are very useful for multi-dimensional signal processing and analysis. In Chapter 5, the basic concepts of quaternion and RB algebra are reviewed, and the author introduce the two dimensional Hermite-Gaussian functions (2D-HGF) as the eigenfunction of DQFT/DRBQFT, and the eigenvalues of 2D-HGF for three types of DQFT and two types of DRBQFT. After that, the relation between 2D-HGF and Gauss-Laguerre circular harmonic function (GLCHF) is given. From the aforementioned relation and some derivations, the GLCHF can be proved as the eigenfunction of DQFT/DRBQFT and its eigenvalues are summarized. These GLCHFs can be used as the basis to perform color image expansion. The expansion coefficients can be used to reconstruct the original color image and as a rotation invariant feature. The GLCHFs can also be applied to color matching applications.

Chapter 6 - The unit quaternion system was invented in 1843 by William Rowan Hamilton as an extension to the complex number to find an answer to the question (how to multiply triplets?). Yet, quaternions are extensively used to represent the attitude of a rigid body such as quadrotors, which is able to alleviate the singularity problem caused by the Euler angles representation. The singularity is in general a point at which a given mathematical object is not defined and it outcome of the so called gimbal lock. The singularity is occur when the pitch angles rotation is $\theta = \pm 90^{\circ}$. In Chapter 6, a leader-follower formation control problem of quadrotors is investigated. The quadrotor dynamic model is represented by unit quaternion with the consideration of external disturbance. Three different control techniques are proposed for both the leader and the follower robots. First, a nonlinear $H_{\infty}$ design approach is derived by solving a Hamilton-Jacobi inequality following from a result for general nonlinear affine systems. Second, integral backstepping (IBS) controllers are also addressed for the leader-follower formation control problem. Then, an iterative Linear Quadratic Regulator (iLQR) is derived to solve the problem of leader-follower formation. The simulation results from all types of controllers are compared and robustness performance of the $H_{\infty}$ controllers, fast convergence and small tracking errors of iLQR controllers over the IBS controllers are demonstrated.

Chapter 7 - A generalized inverse of a given quaternion matrix (similarly, as for complex matrices) exists for a larger class of matrices than the invertible matrices. It has some of the properties of the usual inverse, and agrees with the inverse when a given matrix happens to be invertible. There exist many different generalized inverses. The authors consider determinantal representations of the Drazin and weighted Drazin inverses over the quaternion skew field. Due to the theory of column-row determinants recently introduced by the author, the authors derive determinantal representations of the Drazin inverse for both Hermitian and arbitrary matrices over the quaternion skew field. Using obtained determinantal representations of the Drazin inverse we get explicit representation formulas (analogs of Cramer's rule) for the Drazin inverse solutions of the quaternionic matrix equations $\mathrm{AXB} = \mathrm{D}$ and, consequently, $\mathrm{AX} = \mathrm{D}$, $\mathrm{XB} = \mathrm{D}$ in both cases when $\mathrm{A}$ and $\mathrm{B}$ are Hermitian and arbitrary, where $\mathrm{A}$, $\mathrm{B}$ can be noninvertible matrices of appropriate sizes. The author obtain determinantal representations of solutions of the differential quaternionic matrix equations, $X' + AX = B$ and $X' + XA = B$, where $A$ is noninvertible as well. Also, the authors obtains new determinantal representations of the W-weighted Drazin inverse over the quaternion skew field. The author give determinantal representations of the W-weighted Drazin inverse by using previously obtained determinantal representations of the Drazin inverse, the Moore-Penrose inverse, and the limit representations of the W-weighted Drazin inverse in some special case. Using these determinantal representations of the W-weighted Drazin inverse, the authors derive explicit formulas for determinantal representations of the W-weighted Drazin inverse solutions of the quaternionic matrix equations $WAWX = D$, $XWAW = D$, and $W_1AW_1XW_2BW_2 = D$.

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*Chapter 7*

# Determinantal Representations of the Drazin and W-Weighted Drazin Inverses Over the Quaternion Skew Field with Applications

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**Keywords**: quaternion matrix, generalized inverse, Drazin inverse, weighted Drazin inverse, Moore-Penrose inverse, weighted Moore-Penrose inverse, system of linear equations, Cramer's rule, quaternion matrix equation, generalized inverse solution, least squares solution, Drazin inverse solution, differential matrix equation

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**Abstract**

A generalized inverse of a given quaternion matrix (similarly, as for complex matrices) exists for a larger class of matrices than the invertible matrices. It has some of the properties of the usual inverse, and agrees with the inverse when a given matrix happens to be invertible. There

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exist many different generalized inverses. In this chapter, we consider determinantal representations of the Drazin and weighted Drazin inverses over the quaternion skew field.

Due to the theory of column-row determinants recently introduced by the author, we derive determinantal representations of the Drazin inverse for both Hermitian and arbitrary matrices over the quaternion skew field. Using obtained determinantal representations of the Drazin inverse we get explicit representation formulas (analogs of Cramer's rule) for the Drazin inverse solutions of the quaternionic matrix equations $\mathbf{AXB} = \mathbf{D}$ and, consequently, $\mathbf{AX} = \mathbf{D}$, $\mathbf{XB} = \mathbf{D}$ in both cases when $\mathbf{A}$ and $\mathbf{B}$ are Hermitian and arbitrary, where $\mathbf{A}$, $\mathbf{B}$ can be noninvertible matrices of appropriate sizes. We obtain determinantal representations of solutions of the differential quaternionic matrix equations, $\mathbf{X}' + \mathbf{AX} = \mathbf{B}$ and $\mathbf{X}' + \mathbf{XA} = \mathbf{B}$, where $\mathbf{A}$ is noninvertible as well.

Also, we obtain new determinantal representations of the W-weighted Drazin inverse over the quaternion skew field. We give determinantal representations of the W-weighted Drazin inverse by using previously obtained determinantal representations of the Drazin inverse, the Moore-Penrose inverse, and the limit representations of the W-weighted Drazin inverse in some special case. Using these determinantal representations of the W-weighted Drazin inverse, we derive explicit formulas for determinantal representations of the W-weighted Drazin inverse solutions of the quaternionic matrix equations $\mathbf{WAWX} = \mathbf{D}$, $\mathbf{XWAW} = \mathbf{D}$, and $\mathbf{W}_1\mathbf{AW}_1\mathbf{XW}_2\mathbf{BW}_2 = \mathbf{D}$.

# 1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex number fields, respectively. Throughout the paper, we denote the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \,|\, i^2 = j^2 = k^2 = -1,\, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m\times n}$, and by $\mathbb{H}_r^{m\times n}$ the set of all $m \times n$ matrices over $\mathbb{H}$ with a rank $r$. Let $\mathrm{M}(n, \mathbb{H})$ be the ring of $n \times n$ quaternion matrices and $\mathbf{I}$ be the identity matrix with the appropriate size. For $\mathbf{A} \in \mathbb{H}^{n\times m}$, we denote by $\mathbf{A}^*$, $\mathrm{rank}\,\mathbf{A}$ the conjugate transpose (Hermitian adjoint) matrix and the rank of $\mathbf{A}$. The matrix $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$ is Hermitian if $\mathbf{A}^* = \mathbf{A}$.

The definitions of the generalized inverse matrices can be extended to quaternion matrices as follows.

**Definition 1.1.** *The Moore-Penrose inverse of* $\mathbf{A} \in \mathbb{H}^{m\times n}$*, denoted by* $\mathbf{A}^{\dagger}$*, is the unique matrix* $\mathbf{X} \in \mathbb{H}^{n\times m}$ *satisfying the following equations,*

$$\mathbf{AXA} = \mathbf{A}; \tag{1.1}$$

$$\mathbf{XAX} = \mathbf{X}; \tag{1.2}$$

$$(\mathbf{AX})^{*} = \mathbf{AX}; \tag{1.3}$$

$$(\mathbf{XA})^{*} = \mathbf{XA}. \tag{1.4}$$

**Definition 1.2.** *For* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *with* $k = Ind\,\mathbf{A}$ *the smallest positive number such that* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^{k}$*, the Drazin inverse of* $\mathbf{A}$ *is defined to be the unique matrix* $\mathbf{X}$ *that satisfying (1..2) and the following equations,*

$$\mathbf{AX} = \mathbf{XA}; \tag{1.5}$$

$$\mathbf{A}^{k+1}\mathbf{X} = \mathbf{A}^{k}. \tag{1.6}$$

*It is denoted by* $\mathbf{X} = \mathbf{A}^{D}$*. In particular, when* $Ind\,\mathbf{A} = 1$*, then the matrix* $\mathbf{X}$ *is called the group inverse and is denoted by* $\mathbf{X} = \mathbf{A}^{g}$*.*

If $Ind\,\mathbf{A} = 0$, then $\mathbf{A}$ is invertible, and $\mathbf{A}^{D} \equiv \mathbf{A}^{\dagger} = \mathbf{A}^{-1}$.

Cline and Greville [1] extended the Drazin inverse of a square matrix to a rectangular matrix, which can be generalized to the quaternion algebra as follows.

**Definition 1.3.** *For* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $\mathbf{W} \in \mathbb{H}^{n\times m}$*, the W-weighted Drazin inverse of* $\mathbf{A}$ *with respect to* $\mathbf{W}$ *is the unique solution to equations,*

$$(\mathbf{AW})^{k+1}\mathbf{XW} = (\mathbf{AW})^{k}; \tag{1.7}$$

$$\mathbf{XWAWX} = \mathbf{X}; \tag{1.8}$$

$$\mathbf{AWX} = \mathbf{XWA}, \tag{1.9}$$

*where* $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$*. It is denoted by* $\mathbf{X} = \mathbf{A}_{d,\mathbf{W}}$*.*

The problem of determinantal representation of generalized inverse matrices only recently begun to be decided through the theory of column-row determinants introduced in [2, 3]. The theory of row and column determinants develops the classical approach to a definition of a determinant as an alternating sum of products of elements of a quadratic matrix but with a predetermined order of factors in each summand of a determinant. A determinant of a matrix with noncommutative elements is often called the noncommutative determinant. Unlike other known noncommutative determinants such as determinants of Dieudonné [4], Study [5], Moore [6,7], Chen [8], quasideterminants of Gelfand-Retakh [9], the double determinant built on the theory of the column-row determinants has properties similar to the usual determinant, in particular, it can be expand along arbitrary rows and columns. This property is necessary for determinantal representations of the inverse and generalized inverse matrices. Determinantal representations of the Moore-Penrose inverse, the minimum norm least squares solutions of some quaternion matrix equations over the quaternion skew-field have been obtained in [10, 11]. Determinantal representations of an outer inverse $\mathbf{A}^{(2)}_{T,S}$ has introduced in [12,13] using column-row determinants as well. Recall that an outer inverse of a matrix $\mathbf{A}$ over complex field with prescribed range space $T$ and null space $S$ is a solution of (1..2) with restrictions,

$$\mathcal{R}(\mathbf{X}) = T, \;\; \mathcal{N}(\mathbf{X}) = S.$$

Within the framework of the theory of column-row determinants Song [14] also has gave a determinantal representation of the W-weighted Drazin inverse over the quaternion skew-field using its characterization by an outer inverse $\mathbf{A}^{(2)}_{T,S}$. But, in obtaining of this determinantal representation, auxiliary matrices that different from $\mathbf{A}$ or its powers are needed. In this chapter, we shall obtain new determinantal representations of the Drazin inverse and the W-weighted Drazin inverse of $\mathbf{A} \in \mathbb{H}^{m\times n}$ with respect to $\mathbf{W} \in \mathbb{H}^{n\times m}$ by using only their entries. These determinantal representations of the Drazin and W-weighted Drazin inverse will be used for explicit determinantal representation formulas of the Drazin and W-weighted Drazin inverse solutions of some quaternion matrix equations.

The chapter is organized as follows. We start with some basic concepts and results from the theory of row-column determinants and the theory of quaternion matrices in Section 2. In Section 3, we give the determinantal representations of the Drazin inverse of a Hermitian quaternion matrix in Subsection 3.1 and an arbitrary quaternion matrix in Subsection 3.2. In Section 4, we obtain explicit representation formulas for the Drazin inverse solutions of quaternion matrix equations $\mathbf{AXB} = \mathbf{D}$ and, consequently, $\mathbf{AX} = \mathbf{D}$, and $\mathbf{XB} = \mathbf{D}$. In Subsection 4.1, we consider the case when $\mathbf{A}$, $\mathbf{B}$ are Hermitian, and they are arbitrary in Subsection 4.1. In Section 4.3, we show numerical examples to illustrate the main results. In Section 5, we apply the obtained determinantal representations of the Drazin inverse to solutions of differential matrix equations. In Subsection 5.1, we give a background for quaternion-valued differential equations. In Subsection 5.2, determinantal representations of solutions of the differential matrix equations, $\mathbf{X}' + \mathbf{AX} = \mathbf{B}$ and $\mathbf{X}' + \mathbf{XA} = \mathbf{B}$ are derived, where $\mathbf{A} \in \mathbb{H}^{n\times n}$ is noninvertible. It is demonstrated in an example in Subsection 5.3. In Section 6, we obtain determinantal representations of the W-weighted Drazin inverse by using introduced above determinantal representations of the Drazin inverse in Subsection 6.1, the Moore-Penrose inverse in Subsection 6.2, and the limit representations of the W-weighted Drazin inverse in some special case in Subsection 6.3. In Subsection 6.4, we show a numerical example to illustrate the main result. By using determinantal representations of the W-weighted Drazin inverse obtained in the previous section, we get explicit formulas for determinantal representations of the W-weighted Drazin inverse solutions (analogs of Cramer's rule) of some quaternion matrix equations in Section 7. In Subsection 7.1, we consider the background of the problem of Cramer's rule for the W-weighted Drazin inverse solution. In Subsection 7.2, we obtain explicit representation formulas of the W-weighted Drazin inverse solutions (analogs of Cramer's rule) of the quaternion matrix equations $\mathbf{WAWX} = \mathbf{D}$, $\mathbf{XWAW} = \mathbf{D}$, and $\mathbf{W}_1\mathbf{AW}_1\mathbf{XW}_2\mathbf{BW}_2 = \mathbf{D}$. In Subsection 7.3, we give numerical examples to illustrate the main result.

Facts set forth in Sections 3 and 4 were partly published in [15], in Section 6 were published in [16] and in Section 7 were partly published in [17].

## 2. Preliminaries. Elements of the Theory of the Column and Row Determinants

Suppose $S_n$ is the symmetric group on the set $I_n = \{1, \ldots, n\}$. Through the chapter, we denote $i = 1, \ldots, n$ by $i = \overline{1, n}$.

**Definition 2.1.** *The $i$-th row determinant of* $\mathbf{A} = (a_{ij}) \in \mathrm{M}\,(n, \mathbb{H})$ *is defined* *for all* $i = \overline{1, n}$ *by putting*

$$\begin{aligned}&\mathrm{rdet}_i \mathbf{A} = \\ &\quad \sum_{\sigma \in S_n} (-1)^{n-r} (a_{i\, i_{k_1}} a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i}) \ldots (a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}}), \\ &\quad \sigma = (i\, i_{k_1} i_{k_1+1} \ldots i_{k_1+l_1}) (i_{k_2} i_{k_2+1} \ldots i_{k_2+l_2}) \ldots (i_{k_r} i_{k_r+1} \ldots i_{k_r+l_r}),\end{aligned}$$

*with conditions* $i_{k_2} < i_{k_3} < \ldots < i_{k_r}$ *and* $i_{k_t} < i_{k_t+s}$ *for* $t = \overline{2, r}$ *and* $s = \overline{1, l_t}$.

**Definition 2.2.** *The* $j$*-th column determinant of* $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ *is defined for all* $j = \overline{1, n}$ *by putting*

$$\begin{aligned}&\mathrm{cdet}_j \mathbf{A} = \\ &\quad \sum_{\tau \in S_n} (-1)^{n-r} (a_{j_{k_r} j_{k_r+l_r}} \ldots a_{j_{k_r+1} i_{k_r}}) \ldots (a_{j\, j_{k_1+l_1}} \ldots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j}), \\ &\quad \tau = (j_{k_r+l_r} \ldots j_{k_r+1} j_{k_r}) \ldots (j_{k_2+l_2} \ldots j_{k_2+1} j_{k_2}) \; (j_{k_1+l_1} \ldots j_{k_1+1} j_{k_1} j),\end{aligned}$$

*with conditions,* $j_{k_2} < j_{k_3} < \ldots < j_{k_r}$ *and* $j_{k_t} < j_{k_t+s}$ *for* $t = \overline{2, r}$ *and* $s = \overline{1, l_t}$.

Suppose $\mathbf{A}^{ij}$ denotes the submatrix of $\mathbf{A}$ obtained by deleting both the $i$-th row and the $j$-th column. Let $\mathbf{a}_{.j}$ be the $j$-th column and $\mathbf{a}_{i.}$ be the $i$-th row of $\mathbf{A}$. Suppose $\mathbf{A}_{.j}(\mathbf{b})$ denotes the matrix obtained from $\mathbf{A}$ by replacing its $j$-th column with the column-vector $\mathbf{b}$, and $\mathbf{A}_{i.}(\mathbf{b})$ denotes the matrix obtained from $\mathbf{A}$ by replacing its $i$-th row with the row-vector $\mathbf{b}$.

We note some properties of column and row determinants of a quaternion matrix $\mathbf{A} = (a_{ij})$, where $i \in I_n$, $j \in J_n$ and $I_n = J_n = \{1, \ldots, n\}$. These properties completely have been proved in [2, 3].

**Proposition 2.1.** *If* $b \in \mathbb{H}$*, then* $\mathrm{rdet}_i \mathbf{A}_{i.}(b \cdot \mathbf{a}_{i.}) = b \cdot \mathrm{rdet}_i \mathbf{A}$ *for all* $i = \overline{1, n}$.

**Proposition 2.2.** *If* $b \in \mathbb{H}$*, then* $\mathrm{cdet}_j \mathbf{A}_{.j}(\mathbf{a}_{.j} \cdot b) = \mathrm{cdet}_j \mathbf{A} \cdot b$ *for all* $j = \overline{1, n}$.

**Proposition 2.3.** *If for* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *there exists* $t \in I_n$ *such that* $a_{tj} = b_j + c_j$ *for all* $j = \overline{1, n}$*, then*

$$\begin{aligned}\mathrm{rdet}_i \mathbf{A} &= \mathrm{rdet}_i \mathbf{A}_{t.}(\mathbf{b}) + \mathrm{rdet}_i \mathbf{A}_{t.}(\mathbf{c}), \\ \mathrm{cdet}_i \mathbf{A} &= \mathrm{cdet}_i \mathbf{A}_{t.}(\mathbf{b}) + \mathrm{cdet}_i \mathbf{A}_{t.}(\mathbf{c}),\end{aligned}$$

*where* $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{H}^{1 \times n}$, $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{H}^{1 \times n}$, $i = \overline{1, n}$.

**Proposition 2.4.** *If for* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *there exists* $t \in J_n$ *such that* $a_{i\,t} = b_i + c_i$ *for all* $i = \overline{1, n}$*, then*

$$\begin{aligned} \mathrm{rdet}_j\, \mathbf{A} &= \mathrm{rdet}_j\, \mathbf{A}_{\,.\,t}(\mathbf{b}) + \mathrm{rdet}_j\, \mathbf{A}_{\,.\,t}(\mathbf{c}), \\ \mathrm{cdet}_j\, \mathbf{A} &= \mathrm{cdet}_j\, \mathbf{A}_{\,.\,t}(\mathbf{b}) + \mathrm{cdet}_j \mathbf{A}_{\,.\,t}(\mathbf{c}), \end{aligned}$$

*where* $\mathbf{b} = (b_1, \ldots, b_n)^T \in \mathbb{H}^{n\times 1}$, $\mathbf{c} = (c_1, \ldots, c_n)^T \in \mathbb{H}^{n\times 1}$, $j = \overline{1, n}$.

**Proposition 2.5.** *If* $\mathbf{A}^*$ *is the Hermitian adjoint matrix of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, then* $\mathrm{rdet}_i\, \mathbf{A}^* = \overline{\mathrm{cdet}_i\, \mathbf{A}}$ *for all* $i = \overline{1, n}$.

The following lemmas enable us to expand $\mathrm{rdet}_i\, \mathbf{A}$ by cofactors along the $i$-th row and $\mathrm{cdet}_j \mathbf{A}$ along the $j$-th column respectively for all $i, j = \overline{1, n}$.

**Lemma 2.3.** *Let* $R_{i\,j}$ *be the* $ij$*-th right cofactor of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, that is,* $\mathrm{rdet}_i\, \mathbf{A} = \sum\limits_{j=1}^{n} a_{i\,j} \cdot R_{i\,j}$ *for all* $i = \overline{1, n}$*. Then*

$$R_{i\,j} = \begin{cases} -\mathrm{rdet}_k\, \mathbf{A}^{ii}_{.j}(\mathbf{a}_{.i}), & i \neq j, \quad k = \begin{cases} j, & if \quad i > j; \\ j-1, & if \quad i < j; \end{cases} \\ \mathrm{rdet}_k\, \mathbf{A}^{ii}, & i = j, \quad k = \min\{I_n \setminus i\}, \end{cases} \tag{2.1}$$

*where* $\mathbf{A}^{i\,i}_{.j}(\mathbf{a}_{.\,i})$ *is obtained from* $\mathbf{A}$ *by replacing the* $j$*-th column with the* $i$*-th column, and then by deleting both the* $i$*-th row and column.*

**Lemma 2.4.** *Let* $L_{i\,j}$ *be the* $ij$*-th left cofactor of* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$*, that is,* $\mathrm{cdet}_j\, \mathbf{A} = \sum\limits_{i=1}^{n} L_{i\,j} \cdot a_{i\,j}$ *for all* $j = \overline{1, n}$*. Then*

$$L_{i\,j} = \begin{cases} -\mathrm{cdet}_k\, \mathbf{A}^{jj}_{i.}(\mathbf{a}_{j.}), & i \neq j, \quad k = \begin{cases} i, & if \quad j > i; \\ i-1, & if \quad j < i; \end{cases} \\ \mathrm{cdet}_k\, \mathbf{A}^{ii}, & i = j, \quad k = \min\{J_n \setminus j\}, \end{cases} \tag{2.2}$$

*where* $\mathbf{A}^{jj}_{i.}(\mathbf{a}_{j\,.})$ *is obtained from* $\mathbf{A}$ *by replacing the* $i$*-th row with the* $j$*-th row, and then by deleting both the* $j$*-th row and column.*

The following theorem has a key value in the theory of column-row determinants.

**Theorem 2.5.** *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian, then* $\mathrm{rdet}_1 \mathbf{A} = \cdots = \mathrm{rdet}_n \mathbf{A} = \mathrm{cdet}_1 \mathbf{A} = \cdots = \mathrm{cdet}_n \mathbf{A} \in \mathbb{R}$.

**Remark 2.6.** *Since all column and row determinants of a Hermitian matrix over $\mathbb{H}$ are equal, we can define the determinant of Hermitian $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$ by putting for all $i = \overline{1, n}$,*

$$\det \mathbf{A} := \mathrm{rdet}_i\, \mathbf{A} = \mathrm{cdet}_i\, \mathbf{A}.$$

Properties of the determinant of a Hermitian matrix is completely explored in [3] by its row and column determinants. They can be summarized by the following theorems.

**Theorem 2.7.** *If the $i$-th row of a Hermitian matrix $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$ is replaced with a left linear combination of its other rows, i.e. $\mathbf{a}_{i.} = c_1 \mathbf{a}_{i_1.} + \ldots + c_k \mathbf{a}_{i_k.}$, where $c_l \in \mathbb{H}$ for all $l = \overline{1, k}$ and $i, i_l \in I_n$, then*

$$\mathrm{rdet}_i\, \mathbf{A}_{i\,.}\,(c_1 \mathbf{a}_{i_1.} + \ldots + c_k \mathbf{a}_{i_k.}) = \mathrm{cdet}_i\, \mathbf{A}_{i\,.}\,(c_1 \mathbf{a}_{i_1.} + \ldots + c_k \mathbf{a}_{i_k.}) = 0.$$

**Theorem 2.8.** *If the $j$-th column of a Hermitian matrix $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$ is replaced with a right linear combination of its other columns, i.e. $\mathbf{a}_{.j} = \mathbf{a}_{.j_1} c_1 + \ldots + \mathbf{a}_{.j_k} c_k$, where $c_l \in \mathbb{H}$ for all $l = \overline{1, k}$ and $j, j_l \in J_n$, then*

$$\mathrm{cdet}_j\, \mathbf{A}_{.j}\,(\mathbf{a}_{.j_1} c_1 + \ldots + \mathbf{a}_{.j_k} c_k) = \mathrm{rdet}_j\, \mathbf{A}_{.j}\,(\mathbf{a}_{.j_1} c_1 + \ldots + \mathbf{a}_{.j_k} c_k) = 0.$$

The following theorem on determinantal representations of an inverse matrix of Hermitian follows directly from these properties.

**Theorem 2.9.** *If $\mathbf{A} \in \mathrm{M}\,(n, \mathbb{H})$ is Hermitian, and $\det \mathbf{A} \neq 0$, then there exists a unique right inverse matrix $(R\mathbf{A})^{-1}$ and a unique left inverse matrix $(L\mathbf{A})^{-1}$ of $\mathbf{A}$, where $(R\mathbf{A})^{-1} = (L\mathbf{A})^{-1} =: \mathbf{A}^{-1}$, and they possess the following determinantal representations, respectively,*

$$(R\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\ R_{12} & R_{22} & \cdots & R_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{pmatrix}, \tag{2.3}$$

$$(L\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\ L_{12} & L_{22} & \cdots & L_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ L_{1n} & L_{2n} & \cdots & L_{nn} \end{pmatrix}, \tag{2.4}$$

*where $R_{ij}$, $L_{ij}$ are the right (2.1) and left (2.2) $ij$-th cofactors of $\mathbf{A}$ for all $i, j = \overline{1, n}$.*

**Remark 2.10.** *If* $\det \mathbf{A} = 0$*, we say that a Hermitian quaternion matrix* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is singular because, in this case,* $\mathbf{A}$ *is noninvertible.*

Since principal submatrices of a Hermitian matrix are Hermitian, the principal minor can be defined as the determinant of its principal submatrix by analogy to the commutative case. In [3], we have introduced the rank by principle minors that is the maximal order of a nonzero principal minor of a Hermitian matrix. The following theorem determines a relationship between it and the column rank of a matrix defining as ceiling amount of right-linearly independent columns, and the row rank defining as ceiling amount of left-linearly independent rows.

**Theorem 2.11.** *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian, then its rank by principal minors are equal to its column and row ranks.*

Due to the non-commutativity of quaternions, there are two types of eigenvalues. A quaternion $\lambda$ is said to be a right eigenvalue of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ if $\mathbf{A} \cdot \mathbf{x} = \mathbf{x} \cdot \lambda$, and $\lambda$ is a left eigenvalue if $\mathbf{A} \cdot \mathbf{x} = \lambda \cdot \mathbf{x}$ for some nonzero quaternion column-vector $\mathbf{x} \in \mathbb{H}^n$.

The theory on the left eigenvalues of quaternion matrices has been investigated, in particular, in [18,19,20]. The theory on the right eigenvalues of quaternion matrices is more developed. In particular, we note [21,23,24,25,26,27].

**Proposition 2.6.** *[25] Let* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *be Hermitian. Then* $\mathbf{A}$ *has exactly* $n$ *real right eigenvalues.*

Right and left eigenvalues are in general unrelated [27] but it is not for Hermitian matrices. Suppose $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is Hermitian and $\lambda \in \mathbb{R}$ is its right eigenvalue, then $\mathbf{A} \cdot \mathbf{x} = \mathbf{x} \cdot \lambda = \lambda \cdot \mathbf{x}$. This means that all right eigenvalues of a Hermitian matrix are its left eigenvalues as well. For real left eigenvalues, $\lambda \in \mathbb{R}$, the matrix $\lambda\mathbf{I} - \mathbf{A}$ is Hermitian.

**Definition 2.12.** *If* $\lambda \in \mathbb{R}$*, then for a Hermitian matrix* $\mathbf{A}$ *the polynomial* $p_{\mathbf{A}}(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$ *is said to be the characteristic polynomial of* $\mathbf{A}$.

The roots of the characteristic polynomial of a Hermitian matrix are its real left eigenvalues which are its right eigenvalues as well. We can prove the following theorem by analogy to the commutative case (see, e.g. [28]).

**Theorem 2.13.** *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian, then* $p_{\mathbf{A}}(\lambda) = \lambda^n - d_1\lambda^{n-1} + d_2\lambda^{n-2} - \ldots + (-1)^n d_n$*, where* $d_k$ *is the sum of principle minors of* $\mathbf{A}$ *of order* $k$*,* $1 \le k < n$*, and* $d_n = \det \mathbf{A}$.

# 3. Determinantal Representations of the Drazin Inverse

As one of the important types of generalized inverses of matrices, the Drazin inverses and their applications have well been examined in the literature (see, e.g., [29, 30, 31, 32, 33, 34]). In [35], Stanimirovič and Djordjević have introduced a determinantal representation of the Drazin inverse of a complex matrix based on its full-rank representation. In [36], we obtain determinantal representations of the Drazin inverse of a complex matrix used its limit representation. It allowed to obtain the analogs of Cramer's rule for the Drazin inverse solutions of some matrix equations. In this chapter we extend studies conducted in [36] from the complex field to the quaternion skew field.

## 3.1. Analogues of the Classical Adjoint Matrix for the Drazin Inverse of a Hermitian Matrix

For Hermitian matrices, we apply the method which consists of the theorem on the limit representation of the Drazin inverse, lemmas on rank of matrices and on characteristic polynomial. This method at first has been used in [36], afterwards in [37, 38]. By analogy to [39] the following theorem on the limit representation of the quaternion Drazin inverse can be proved.

**Theorem 3.1.** *If $\mathbf{A} \in \mathbb{H}^{n\times n}$ with $Ind\,\mathbf{A} = k$, then*

$$\mathbf{A}^{D} = \lim_{\lambda\to 0}\left(\lambda\mathbf{I}_n + \mathbf{A}^{k+1}\right)^{-1}\mathbf{A}^{k} = \lim_{\lambda\to 0}\mathbf{A}^{k}\left(\lambda\mathbf{I}_n + \mathbf{A}^{k+1}\right)^{-1},$$

*where $\lambda \in \mathbb{R}_+$, and $\mathbb{R}_+$ is a set of the real positive numbers.*

Denote by $\mathbf{a}_{.j}^{(m)}$ and $\mathbf{a}_{i.}^{(m)}$ the $j$-th column and the $i$-th row of $\mathbf{A}^m$, respectively.

**Lemma 3.2.** *If $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$ with $Ind\,\mathbf{A} = k$, then*

$$\operatorname{rank}\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right) \leq \operatorname{rank}\left(\mathbf{A}^{k+1}\right). \tag{3.1}$$

*Proof.* The matrix $\mathbf{A}_{i\,.}^{k+1}\left(\mathbf{a}_{j\,.}^{(k)}\right)$ can be represented as follows

$$\mathbf{A}_{i\,.}^{k+1}\left(\mathbf{a}_{j\,.}^{(k)}\right)=\begin{pmatrix} \sum\limits_{s=1}^{n} a_{1s}a_{s1}^{(k)} & \dots & \sum\limits_{s=1}^{n} a_{1s}a_{sn}^{(k)} \\ \dots & \dots & \dots \\ a_{j1}^{(k)} & \dots & a_{j\,n}^{(k)} \\ \dots & \dots & \dots \\ \sum\limits_{s=1}^{n} a_{ns}a_{s1}^{(k)} & \dots & \sum\limits_{s=1}^{n} a_{ns}a_{sn}^{(k)} \end{pmatrix}$$

Let $\mathbf{P}_{l\,i}\left(-a_{l\,j}\right)\in\mathbb{H}^{n\times n}$, $(l\neq i)$, be a matrix with $-a_{l\,j}$ in the $(l,i)$-entry, 1 in all diagonal entries, and 0 in others. This is a matrix of an elementary transformation. It follows that

$$\tilde{\mathbf{A}}:=\mathbf{A}_{i\,.}^{k+1}\left(\mathbf{a}_{j\,.}^{(k)}\right)\cdot\prod_{l\neq i}\mathbf{P}_{l\,i}\left(-a_{l\,j}\right)=\begin{pmatrix} \sum\limits_{s\neq j} a_{1s}a_{s1}^{(k)} & \dots & \sum\limits_{s\neq j} a_{1s}a_{sn}^{(k)} \\ \dots & \dots & \dots \\ a_{j1}^{(k)} & \dots & a_{j\,n}^{(k)} \\ \dots & \dots & \dots \\ \sum\limits_{s\neq j} a_{ns}a_{s1}^{(k)} & \dots & \sum\limits_{s\neq j} a_{ns}a_{sn}^{(k)} \end{pmatrix} i{-}th$$

The above obtained matrix $\tilde{\mathbf{A}}$ has the following factorization.

$$\tilde{\mathbf{A}}=\begin{pmatrix} a_{11} & \dots & 0 & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & 0 & \dots & a_{nn} \end{pmatrix}\begin{pmatrix} a_{11}^{(k)} & a_{12}^{(k)} & \dots & a_{1n}^{(k)} \\ a_{21}^{(k)} & a_{22}^{(k)} & \dots & a_{2n}^{(k)} \\ \dots & \dots & \dots & \dots \\ a_{n1}^{(k)} & a_{n2}^{(k)} & \dots & a_{nn}^{(k)} \end{pmatrix}$$

Denote the first matrix by

$$\tilde{\mathbf{A}}_1:=\begin{pmatrix} a_{11} & \dots & 0 & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & \underset{j-th}{0} & \dots & a_{nn} \end{pmatrix} i-th.$$

The matrix $\tilde{\mathbf{A}}_1$ is obtained from $\mathbf{A}$ by replacing all entries of the $i$-th row and the $j$-th column with zeroes except for 1 in the $(i,j)$-entry. Since elementary transformations of a matrix do not change its rank, then $\operatorname{rank} \mathbf{A}_{i\,.}^{k+1}\left(\mathbf{a}_{j\,.}^{(k)}\right) \leq \min\left\{\operatorname{rank} \mathbf{A}^k, \operatorname{rank} \tilde{\mathbf{A}}\right\}$. By $\operatorname{rank} \tilde{\mathbf{A}}_1 \geq \operatorname{rank} \mathbf{A}^k$, the proof is completed. □

The next lemma is proved similarly.

**Lemma 3.3.** *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *with* $Ind\,\mathbf{A} = k$*, then* $\operatorname{rank}\left(\mathbf{A}^{k+1}\right)_{i\,.}\left(\mathbf{a}_{j\,.}^{(k)}\right) \leq \operatorname{rank}\left(\mathbf{A}^{k+1}\right)$.

We shall use the following notations. Let $\alpha := \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\beta := \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\}$ be subsets of the order $1 \leq k \leq \min\{m, n\}$. By $\mathbf{A}_\beta^\alpha$ denote the submatrix of $\mathbf{A}$ determined by rows indexed by $\alpha$ and columns indexed by $\beta$. Then $\mathbf{A}_\alpha^\alpha$ denotes the principal submatrix determined by the rows and columns indexed by $\alpha$. If $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is Hermitian, then by $|\mathbf{A}_\alpha^\alpha|$ denote the corresponding principal minor of $\det \mathbf{A}$. For $1 \leq k \leq n$, the collection of strictly increasing sequences of $k$ integers chosen from $\{1, \ldots, n\}$ is denoted by $L_{k,n} := \{\alpha : \alpha = (\alpha_1, \ldots, \alpha_k),\ 1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq n\}$. For fixed $i \in \alpha$ and $j \in \beta$, let $I_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}$, $\quad J_{r,n}\{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\}$.

Analogues of the characteristic polynomial are considered in the following two lemmas.

**Lemma 3.4.** *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian with* $Ind\,\mathbf{A} = k$ *and* $\lambda \in \mathbb{R}$*, then*

$$\operatorname{cdet}_i\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.j}^{(k)}\right) = c_1^{(ij)}\lambda^{n-1} + c_2^{(ij)}\lambda^{n-2} + \ldots + c_n^{(ij)}, \tag{3.2}$$

*where* $c_n^{(ij)} = \operatorname{cdet}_i\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.j}^{(k)}\right)$ *and*

$$c_s^{(ij)} = \sum_{\beta \in J_{s,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.j}^{k}\right)\right)_\beta^\beta$$

*for all* $s = \overline{1, n-1}$*,* $i, j = \overline{1, n}$*.*

*Proof.* Denote by $\mathbf{b}_{.\,i}$ the $i$-th column of $\mathbf{A}^{k+1} =: (b_{ij})_{n\times n}$. Consider the Hermitian matrix $\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)_{.i}(\mathbf{b}_{.i}) \in \mathbb{H}^{n\times n}$. It differs from $\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)$ in an entry $b_{ii}$. Taking into account Theorem 2.13, we obtain

$$\det\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)_{.i}(\mathbf{b}_{.\,i}) = d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_n, \tag{3.3}$$

where $d_s = \sum\limits_{\beta \in J_{s,n}\{i\}} |\left(\mathbf{A}^{k+1}\right)_{\beta}^{\beta}|$ is the sum of all principal minors of order $s$ that contain the $i$-th column for all $s = \overline{1, n-1}$ and $d_n = \det\left(\mathbf{A}^{k+1}\right)$.

Consequently, we have $\mathbf{b}_{.i} = \begin{pmatrix} \sum\limits_l a_{1l}^{(k)} a_{li} \\ \sum\limits_l a_{2l}^{(k)} a_{li} \\ \vdots \\ \sum\limits_l a_{nl}^{(k)} a_{li} \end{pmatrix} = \sum\limits_l \mathbf{a}_{.l}^{(k)} a_{li}$, where $\mathbf{a}_{.l}^{(k)}$ is the $l$-th column of $\mathbf{A}^k$ for all $l = \overline{1, n}$. Due to Theorem 2.5, Lemma 2.4 and Proposition 2.2, we obtain on the one hand

$$\det\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)_{.i}(\mathbf{b}_{.i}) = \operatorname{cdet}_i\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)_{.i}(\mathbf{b}_{.i}) =$$
$$\sum_l \operatorname{cdet}_i\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)_{.l}\left(\mathbf{a}_{.l}^{(k)} a_{l\,i}\right) =$$
$$\sum_l \operatorname{cdet}_i\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.l}^{(k)}\right) \cdot a_{li}. \tag{3.4}$$

On the other hand having changed the order of summation, for all $s = \overline{1, n-1}$ we have

$$d_s = \sum_{\beta \in J_{s,n}\{i\}} \det\left(\mathbf{A}^{k+1}\right)_{\beta}^{\beta} = \sum_{\beta \in J_{s,n}\{i\}} \operatorname{cdet}_i\left(\mathbf{A}^{k+1}\right)_{\beta}^{\beta} =$$
$$\sum_{\beta \in J_{s,n}\{i\}} \sum_l \operatorname{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.l}^{(k)} a_{l\,i}\right)\right)_{\beta}^{\beta} =$$
$$\sum_l \sum_{\beta \in J_{s,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.l}^{(k)}\right)\right)_{\beta}^{\beta} \cdot a_{l\,i}. \tag{3.5}$$

By substituting (3.4) and (3.5) in (3.3), and equating factors at $a_{l\,i}$ when $l = j$, we obtain (3.2). □

The following lemma can be proved similarly.

**Lemma 3.5.** *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian with* $Ind\,\mathbf{A} = k$ *and* $\lambda \in \mathbb{R}$, *then*

$$\operatorname{rdet}_j(\lambda\mathbf{I} + \mathbf{A}^{k+1})_{j\,.}(\mathbf{a}_{i.}^{(k)}) = r_1^{(ij)}\lambda^{n-1} + r_2^{(ij)}\lambda^{n-2} + \ldots + r_n^{(ij)},$$

*where* $r_n^{(ij)} = \mathrm{rdet}_j(\mathbf{A}^{k+1})_{j.}(\mathbf{a}_{i.}^{(k)})$ *and* $r_s^{(ij)} = \sum\limits_{\alpha\in I_{s,n}\{j\}} \mathrm{rdet}_j\left((\mathbf{A}^{k+1})_{j.}(\mathbf{a}_{i.}^{(k)})\right)_{\alpha}^{\alpha}$ *for all* $s = \overline{1, n-1}$ *and* $i, j = \overline{1, n}$.

**Theorem 3.6.** *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *is Hermitian with* $Ind\,\mathbf{A} = k$ *and* $\mathrm{rank}\,\mathbf{A}^{k+1} = \mathrm{rank}\,\mathbf{A}^k = r$*, then the Drazin inverse* $\mathbf{A}^D = \left(a_{ij}^D\right) \in \mathbb{H}^{n\times n}$ *possess the following determinantal representations:*

$$a_{ij}^D = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{k}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{A}^{k+1}\right)_{\beta}^{\beta}\right|}, \tag{3.6}$$

*or*

$$a_{ij}^D = \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \mathrm{rdet}_j\left((\mathbf{A}^{k+1})_{j.}(\mathbf{a}_{i.}^{(k)})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,n}} \left|(\mathbf{A}^{k+1})_{\alpha}^{\alpha}\right|}. \tag{3.7}$$

*Proof.* At first we prove (3.6). By Theorem 3.1, $\mathbf{A}^D = \lim\limits_{\lambda\to 0}\left(\lambda\mathbf{I}_n + \mathbf{A}^{k+1}\right)^{-1}\mathbf{A}^k$. The matrix $\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right) \in \mathbb{H}^{n\times n}$ is a full-rank Hermitian matrix. Taking into account Theorem 2.9, it has an inverse which can be represented as a left inverse,

$$\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)^{-1} = \frac{1}{\det\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)}\begin{pmatrix} L_{11} & L_{21} & \ldots & L_{n1} \\ L_{12} & L_{22} & \ldots & L_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ L_{1n} & L_{2n} & \ldots & L_{nn} \end{pmatrix},$$

where $L_{ij}$ is a left $ij$-th cofactor of a matrix $\lambda\mathbf{I} + \mathbf{A}^{k+1}$. Then, we have

$$\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)^{-1}\mathbf{A}^k =$$
$$= \frac{1}{\det\left(\lambda\mathbf{I} + \mathbf{A}^{k+1}\right)}\begin{pmatrix} \sum\limits_{s=1}^{n} L_{s1}a_{s1}^{(k)} & \sum\limits_{s=1}^{n} L_{s1}a_{s2}^{(k)} & \ldots & \sum\limits_{s=1}^{n} L_{s1}a_{sn}^{(k)} \\ \sum\limits_{s=1}^{n} L_{s2}a_{s1}^{(k)} & \sum\limits_{s=1}^{n} L_{s2}a_{s2}^{(k)} & \ldots & \sum\limits_{s=1}^{n} L_{s2}a_{sn}^{(k)} \\ \ldots & \ldots & \ldots & \ldots \\ \sum\limits_{s=1}^{n} L_{sn}a_{s1}^{(k)} & \sum\limits_{s=1}^{n} L_{sn}a_{s2}^{(k)} & \ldots & \sum\limits_{s=1}^{n} L_{sn}a_{sn}^{(k)} \end{pmatrix}.$$

By using the definition of a left cofactor, we obtain

$$\mathbf{A}^{D}=\lim_{\lambda\to 0}\begin{pmatrix} \frac{\mathrm{cdet}_1\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{.1}\left(\mathbf{a}_{.1}^{(k)}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)} & \ldots & \frac{\mathrm{cdet}_1\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{.1}\left(\mathbf{a}_{.n}^{(k)}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)} \\ \ldots & \ldots & \ldots \\ \frac{\mathrm{cdet}_n\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{.n}\left(\mathbf{a}_{.1}^{(k)}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)} & \ldots & \frac{\mathrm{cdet}_n\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{.n}\left(\mathbf{a}_{.n}^{(k)}\right)}{\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)} \end{pmatrix}. \tag{3.8}$$

By Theorem 2.13, we have

$$\det\left(\lambda\mathbf{I}+\mathbf{A}^{m+1}\right)=\lambda^{n}+d_1\lambda^{n-1}+d_2\lambda^{n-2}+\ldots+d_n,$$

where $d_s=\sum_{\beta\in J_{s,\,n}}\left|\left(\mathbf{A}^{k+1}\right)_{\beta}^{\beta}\right|$ is a sum of principal minors of $\mathbf{A}^{k+1}$ of order $s$ for all $s=\overline{1,n-1}$ and $d_n=\det\mathbf{A}^{k+1}$.

Since $\operatorname{rank}\mathbf{A}^{k+1}=\operatorname{rank}\mathbf{A}^{k}=r$, then $d_n=d_{n-1}=\ldots=d_{r+1}=0$. It follows that $\det\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)=\lambda^{n}+d_1\lambda^{n-1}+d_2\lambda^{n-2}+\ldots+d_r\lambda^{n-r}$.

Using (3.2), we have

$$\mathrm{cdet}_i\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.j}^{(k)}\right)=c_1^{(ij)}\lambda^{n-1}+c_2^{(ij)}\lambda^{n-2}+\ldots+c_n^{(ij)}$$

for all $i,j=\overline{1,n}$, where $c_s^{(ij)}=\sum_{\beta\in J_{s,\,n}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$ for all $s=\overline{1,n-1}$ and $c_n^{(ij)}=\mathrm{cdet}_i\left(\mathbf{A}^{k+1}\right)_{.i}\left(\mathbf{a}_{.j}^{(k)}\right)$. We shall prove that $c_k^{(ij)}=0$, when $k\geq r+1$ for all $i,j=\overline{1,n}$.

Since by Lemma 3.2, $\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\leq r$, then the matrix $\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)$ has no more $r$ right-linearly independent columns. Consider $\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$, when $\beta\in J_{s,n}\{i\}$. This is a principal submatrix of $\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)$ of order $s\geq r+1$. Deleting both its $i$-th row and column, we obtain a principal submatrix of order $s-1$ of $\mathbf{A}^{k+1}$. We denote it by $\mathbf{M}$. The following cases are possible.

- Let $s=r+1$ and $\det\mathbf{M}\neq 0$. In this case all columns of $\mathbf{M}$ are right-linearly independent. The addition of all of them on one coordinate to columns of $\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$ keeps their right-linear independence.

Hence, they are basis in the matrix $\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$, and the $i$-th column is the right linear combination of its basis columns. From this by Theorem 2.8, we get $\mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}=0$, when $\beta\in J_{s,n}\{i\}$ and $s=r+1$.

- If $s=r+1$ and $\det\mathbf{M}=0$, then $p$, ($p<s$), columns are basis in $\mathbf{M}$ and in $\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$. So, by Theorems 2.11 and 2.8 we obtain $\mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}=0$ as well.
- If $s>r+1$, then from Theorem 2.11 it follows that $\det\mathbf{M}=0$ and $p$, ($p<r$), columns are basis in the both matrices $\mathbf{M}$ and $\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$. Therefor, by Theorem 2.8, we have $\mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}=0$.

Thus, in all cases, $\mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}=0$, when $\beta\in J_{s,n}\{i\}$ and $r+1\leq s<n$. From here, if $r+1\leq s<n$, then

$$c_s^{(ij)}=\sum_{\beta\in J_{s,\,n}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}=0,$$

and $c_n^{(ij)}=\mathrm{cdet}_i\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)=0$ for all $i,j=\overline{1,n}$.

Hence, $\mathrm{cdet}_i\left(\lambda\mathbf{I}+\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.\,j}^{(k)}\right)=c_1^{(ij)}\lambda^{n-1}+c_2^{(ij)}\lambda^{n-2}+\ldots+c_r^{(ij)}\lambda^{n-r}$ for all $i,j=\overline{1,n}$. By substituting these values in the matrix from (3.8), we obtain

$$\mathbf{A}^D=\lim_{\lambda\to 0}\begin{pmatrix}\frac{c_1^{(11)}\lambda^{n-1}+\ldots+c_r^{(11)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} & \ldots & \frac{c_1^{(1n)}\lambda^{n-1}+\ldots+c_r^{(1n)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}}\\ \ldots & \ldots & \ldots\\ \frac{c_1^{(n1)}\lambda^{n-1}+\ldots+c_r^{(n1)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} & \ldots & \frac{c_1^{(nn)}\lambda^{n-1}+\ldots+c_r^{(nn)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}}\end{pmatrix}=$$

$$\begin{pmatrix}\frac{c_r^{(11)}}{d_r} & \ldots & \frac{c_r^{(1n)}}{d_r}\\ \ldots & \ldots & \ldots\\ \frac{c_r^{(n1)}}{d_r} & \ldots & \frac{c_r^{(nn)}}{d_r}\end{pmatrix},$$

where $c_r^{(ij)} = \sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k)}\right)\right){}_{\beta}^{\beta}$ and $d_r = \sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{A}^{k+1}\right){}_{\beta}^{\beta}\right|$.

Thus, we obtained the determinantal representation of $\mathbf{A}^D$ by (3.6).

The determinantal representation of $\mathbf{A}^D$ by (3.7) can be proved similarly. □

In the following corollaries we introduce determinantal representations of the group inverse $\mathbf{A}^g$ and the projection matrices $\mathbf{A}^D\mathbf{A}$ and $\mathbf{A}\mathbf{A}^D$, respectively.

**Corollary 3.1.** *If* $Ind\,\mathbf{A} = 1$ *and* $\operatorname{rank}\mathbf{A}^2 = \operatorname{rank}\mathbf{A} = r \leq n$ *for a Hermitian matrix* $\mathbf{A} \in \mathbb{H}^{n\times n}$*, then the group inverse* $\mathbf{A}^g = \left(a_{ij}^g\right)_{n\times n}$ *possess the following determinantal representations:*

$$a_{ij}^g = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^2\right)_{.\,i}\left(\mathbf{a}_{.j}\right)\right){}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{A}^2\right){}_{\beta}^{\beta}\right|},$$

*or*

$$a_{ij}^g = \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{A}^2\right)_{j\,.}\left(\mathbf{a}_{i.}\right)\right){}_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,n}} \left|\left(\mathbf{A}^2\right){}_{\alpha}^{\alpha}\right|}.$$

*Proof.* The proof follows immediately from Theorem 3.6 in view of $k = 1$. □

**Corollary 3.2.** *If* $Ind\,\mathbf{A} = k$ *and* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^k = r \leq n$ *for a Hermitian matrix* $\mathbf{A} \in \mathbb{H}^{n\times n}$*, then*

$$\mathbf{A}^D\mathbf{A} = \left(\frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k+1}\right)_{.\,i}\left(\mathbf{a}_{.j}^{(k+1)}\right)\right){}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{A}^{k+1}\right){}_{\beta}^{\beta}\right|}\right)_{n\times n}, \tag{3.9}$$

*and*

$$\mathbf{A}\mathbf{A}^D = \left(\frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{A}^{k+1}\right)_{j\,.}\left(\mathbf{a}_{i.}^{(k+1)}\right)\right){}_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,n}} \left|\left(\mathbf{A}^{k+1}\right){}_{\alpha}^{\alpha}\right|}\right)_{n\times n}. \tag{3.10}$$

*Proof.* At first we prove (3.9). Let $\mathbf{A}^D\mathbf{A} = (v_{ij})_{n\times n}$. Using (3.6) for all $i, j = \overline{1, n}$, we have

$$v_{i\,j} = \sum_s \frac{\sum\limits_{\beta \in J_{r,\,n}\{i\}} \operatorname{cdet}_i \left( \left(\mathbf{A}^{k+1}\right)_{.\,i} \left(\mathbf{a}_{.j}^{(k)}\right)\right) {}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left(\mathbf{A}^{k+1}\right) {}_{\beta}^{\beta} \right|} \cdot a_{s\,j} =$$

$$\frac{\sum\limits_{\beta \in J_{r,\,n}\{i\}} \sum\limits_s \operatorname{cdet}_i \left( \left(\mathbf{A}^{k+1}\right)_{.\,i} \left(\mathbf{a}_{.j}^{(k)} \cdot a_{s\,j}\right)\right) {}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left(\mathbf{A}^{k+1}\right) {}_{\beta}^{\beta} \right|} = \frac{\sum\limits_{\beta \in J_{r,\,n}\{i\}} \operatorname{cdet}_i \left( \left(\mathbf{A}^{k+1}\right)_{.\,i} \left(\mathbf{a}_{.j}^{(k+1)}\right)\right) {}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left(\mathbf{A}^{k+1}\right) {}_{\beta}^{\beta} \right|}.$$

By analogy can be proved (3.10), using the determinantal representation of the Drazin inverse by (3.7). □

### 3.2. Determinantal Representations of the Drazin Inverse for an Arbitrary Matrix

For an arbitrary matrix $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ with $Ind\,\mathbf{A} = k$ and $\operatorname{rank} \mathbf{A}^{k+1} = \operatorname{rank} \mathbf{A}^k = r$, we can not apply the method proposed for Hermitian matrices primarily because the lemma on the characteristic polynomial for an arbitrary quaternion matrix is not possible in general. We shall use a basic equality on the Drazin inverse and determinantal representations of the Moore-Penrose inverse by the following proposition and theorem, respectively.

**Proposition 3.1.** *[30] If $Ind(\mathbf{A}) = k$, then*

$$\mathbf{A}^D = \mathbf{A}^k (\mathbf{A}^{2k+1})^+ \mathbf{A}^k.$$

**Theorem 3.7.** *[10] If $\mathbf{A} \in \mathbb{H}_r^{m\times n}$, then the Moore-Penrose inverse $\mathbf{A}^+ = \left(a_{ij}^+\right) \in \mathbb{H}^{n\times m}$ possess the following determinantal representations:*

$$a_{ij}^+ = \frac{\sum\limits_{\beta \in J_{r,\,n}\{i\}} \operatorname{cdet}_i \left( \left(\mathbf{A}^*\mathbf{A}\right)_{.\,i} \left(\mathbf{a}_{.j}^*\right)\right) {}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left(\mathbf{A}^*\mathbf{A}\right) {}_{\beta}^{\beta} \right|}, \tag{3.11}$$

*or*

$$a_{ij}^+ = \frac{\sum\limits_{\alpha \in I_{r,m}\{j\}} \operatorname{rdet}_j \left( (\mathbf{A}\mathbf{A}^*)_{j\,.} (\mathbf{a}_{i.}^*) \right) {}_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{r,\,m}} \left| (\mathbf{A}\mathbf{A}^*) {}_{\alpha}^{\alpha} \right|}, \tag{3.12}$$

*for all* $i = \overline{1,n}$, $j = \overline{1,m}$.

Therefore, an entry of the Drazin inverse of $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is

$$a_{ij}^{D} = \sum_{s=1}^{n} \sum_{t=1}^{n} a_{it}^{(k)} \left( a_{ts}^{(2k+1)} \right)^{+} a_{sj}^{(k)} \tag{3.13}$$

for all $i, j = \overline{1,n}$. Denote by $\hat{\mathbf{a}}_{.s}$ the $s$-th column of $(\mathbf{A}^{2k+1})^{*}\mathbf{A}^{k} =: \hat{\mathbf{A}} = (\hat{a}_{ij}) \in \mathbb{H}^{n\times n}$ for all $s = \overline{1,n}$. It follows from $\sum\limits_{s} \left( \mathbf{a}_{.s}^{(2k+1)} \right)^{*} a_{sj}^{(k)} = \hat{\mathbf{a}}_{.j}$ and (3.11) that

$$\sum_{s=1}^{n} \left( a_{ts}^{(2k+1)} \right)^{+} a_{sj}^{(k)} =$$

$$\sum_{s=1}^{n} \frac{\sum\limits_{\beta \in J_{r,\,n}\{t\}} \operatorname{cdet}_{t} \left( \left( \left( \mathbf{A}^{2k+1} \right)^{*} \left( \mathbf{A}^{2k+1} \right) \right)_{.\,t} \left( \mathbf{a}_{.s}^{(2k+1)} \right)^{*} \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left( \mathbf{A}^{2k+1} \right)^{*} \left( \mathbf{A}^{2k+1} \right) {}_{\beta}^{\beta} \right|} \cdot a_{sj}^{(k)} =$$

$$\frac{\sum\limits_{\beta \in J_{r,\,n}\{t\}} \operatorname{cdet}_{t} \left( \left( \mathbf{A}^{2k+1} \right)^{*} \left( \mathbf{A}^{2k+1} \right)_{.\,t} \left( \hat{\mathbf{a}}_{.\,j} \right) \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left( \mathbf{A}^{2k+1} \right)^{*} \left( \mathbf{A}^{2k+1} \right) {}_{\beta}^{\beta} \right|}.$$

So, the Drazin inverse $\mathbf{A}^{D}$ possess the following determinantal representation,

$$a_{ij}^{D} = \frac{\sum\limits_{t=1}^{n} a_{it}^{(k)} \sum\limits_{\beta \in J_{r,\,n}\{t\}} \operatorname{cdet}_{t} \left( \left( \mathbf{A}^{2k+1} \right)^{*} \left( \mathbf{A}^{2k+1} \right)_{.\,t} \left( \hat{\mathbf{a}}_{.\,j} \right) \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left( \mathbf{A}^{2k+1} \right)^{*} \left( \mathbf{A}^{2k+1} \right) {}_{\beta}^{\beta} \right|}, \tag{3.14}$$

for all $i, j = \overline{1,n}$.

Denote by $\check{\mathbf{a}}_{t.}$ the $t$-th row of $\mathbf{A}^{k}(\mathbf{A}^{2k+1})^{*} =: \check{\mathbf{A}} = (\check{a}_{ij}) \in \mathbb{H}^{n\times n}$ for all $t = \overline{1,n}$. It follows from $a_{it}^{(k)} \sum\limits_{t} \left( \mathbf{a}_{t\,.}^{(2k+1)} \right)^{*} = \check{\mathbf{a}}_{i\,.}$ and (3.11) that

$$\sum_{t=1}^{n} a_{it}^{(k)} \left(a_{ts}^{(2k+1)}\right)^{+} = \sum_{t=1}^{n} a_{it}^{(k)} \cdot \frac{\sum\limits_{\alpha \in I_{r,\,n}\{s\}} \operatorname{rdet}_s \left(\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^{*}\right)_{.s} \left(\mathbf{a}_{.t}^{(2k+1)}\right)^{*}\right) {}_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{r,\,n}} \left|\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^{*}\right) {}_{\alpha}^{\alpha}\right|} =$$

$$\frac{\sum\limits_{\alpha \in I_{r,\,n}\{s\}} \operatorname{rdet}_s \left(\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^{*}\right)_{.s} \left(\check{\mathbf{a}}_{i\,.}\right)\right) {}_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{r,\,n}} \left|\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^{*}\right) {}_{\alpha}^{\alpha}\right|}$$

Therefore, the Drazin inverse $\mathbf{A}^D$ possess the following determinantal representation,

$$a_{ij}^{D} = \frac{\sum\limits_{s=1}^{n} \left( \sum\limits_{\alpha \in I_{r,\,n}\{s\}} \operatorname{rdet}_s \left(\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^{*}\right)_{.s} \left(\check{\mathbf{a}}_{i\,.}\right)\right) {}_{\alpha}^{\alpha} \right) a_{sj}^{(k)}}{\sum\limits_{\alpha \in I_{r,\,n}} \left|\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^{*}\right) {}_{\alpha}^{\alpha}\right|}, \qquad (3.15)$$

for all $i, j = \overline{1, n}$. Thus, we have proved the following theorem.

**Theorem 3.8.** *If* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$ *with* $Ind\,\mathbf{A} = k$ *and* $\operatorname{rank} \mathbf{A}^{k+1} = \operatorname{rank} \mathbf{A}^{k} = r$*, then the Drazin inverse* $\mathbf{A}^D$ *possess the determinantal representations (3.14) and (3.15).*

Using obtained determinantal representations (3.14) and (3.15), we have the following corollaries. Their proofs are similarly to the proofs of Corollaries **??** and **??**, respectively.

**Corollary 3.3.** *If* $\mathbf{A} \in \mathrm{M}\left(n, \mathbb{H}\right)$ *with* $Ind\,\mathbf{A} = 1$ *and* $\operatorname{rank} \mathbf{A}^{2} = \operatorname{rank} \mathbf{A}^{2} = r$*, then the group inverse* $\mathbf{A}^{g} = \left(a_{ij}^{g}\right)_{n \times n}$ *possess the following determinantal representations*

$$a_{ij}^{g} = \frac{\sum\limits_{t=1}^{n} a_{it} \sum\limits_{\beta \in J_{r,\,n}\{t\}} \operatorname{cdet}_t \left(\left(\mathbf{A}^{3}\right)^{*}\left(\mathbf{A}^{3}\right)_{.\,t} \left(\hat{\mathbf{a}}_{.\,j}\right)\right) {}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left|\left(\mathbf{A}^{3}\right)^{*}\left(\mathbf{A}^{3}\right) {}_{\beta}^{\beta}\right|},$$

$$a_{ij}^{g} = \frac{\sum\limits_{s=1}^{n} \left( \sum\limits_{\alpha \in I_{r,\,n}\{s\}} \operatorname{rdet}_s \left(\left(\mathbf{A}^{3}\left(\mathbf{A}^{3}\right)^{*}\right)_{.s} \left(\check{\mathbf{a}}_{i\,.}\right)\right) {}_{\alpha}^{\alpha} \right) a_{sj}}{\sum\limits_{\alpha \in I_{r,\,n}} \left|\left(\mathbf{A}^{3}\left(\mathbf{A}^{3}\right)^{*}\right) {}_{\alpha}^{\alpha}\right|},$$

*for all* $i, j = \overline{1, n}$.

**Corollary 3.4.** *If* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *with* $Ind\,\mathbf{A} = k$ *and* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^{k} = r$*, then*

$$\mathbf{A}^D\mathbf{A} = \left( \frac{\sum\limits_{s=1}^{n} \left( \sum\limits_{\alpha \in I_{r,\,n}\{s\}} \operatorname{rdet}_s \left( \left( \mathbf{A}^{2k+1} \left( \mathbf{A}^{2k+1} \right)^* \right)_{.s} (\check{\mathbf{a}}_{i\,.}) \right)_{\alpha}^{\alpha} \right) a_{sj}^{(k+1)}}{\sum\limits_{\alpha \in I_{r,\,n}} \left| \left( \mathbf{A}^{2k+1} \left( \mathbf{A}^{2k+1} \right)^* \right)_{\alpha}^{\alpha} \right|} \right),$$

*and*

$$\mathbf{A}\mathbf{A}^D = \left( \frac{\sum\limits_{t=1}^{n} a_{it}^{(k+1)} \sum\limits_{\beta \in J_{r,\,n}\{t\}} \operatorname{cdet}_t \left( \left( \mathbf{A}^{2k+1} \right)^* \left( \mathbf{A}^{2k+1} \right)_{.\,t} (\hat{\mathbf{a}}_{.\,j}) \right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left( \mathbf{A}^{2k+1} \right)^* \left( \mathbf{A}^{2k+1} \right)_{\beta}^{\beta} \right|} \right),$$

*where* $\mathbf{A}^k(\mathbf{A}^{2k+1})^* = \check{\mathbf{A}} = (\check{a}_{ij})$ *and* $(\mathbf{A}^{2k+1})^*\mathbf{A}^k = \hat{\mathbf{A}} = (\hat{a}_{ij})$.

## 4. Cramer's Rule of the Drazin Inverse Solutions of Some Matrix Equations

One of the main applications of the determinantal representation of an inverse matrix by the classical adjoint matrix is the Cramer rule. In this section we shall show that the obtained determinantal representations give the exact analogues of Cramer's rule for the Drazin inverse solutions of some matrix equations.

For an arbitrary matrix over the quaternion skew field, $\mathbf{A} \in \mathbb{H}^{m \times n}$, we denote by

- $\mathcal{R}_r(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^m : \ \mathbf{y} = \mathbf{A}\mathbf{x}, \ \mathbf{x} \in \mathbb{H}^n\}$, the column right space of $\mathbf{A}$,
- $\mathcal{N}_r(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^n : \ \mathbf{A}\mathbf{x} = 0\}$, the right null space of $\mathbf{A}$,
- $\mathcal{R}_l(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^n : \ \mathbf{y} = \mathbf{x}\mathbf{A}, \ \mathbf{x} \in \mathbb{H}^m\}$, the column left space of $\mathbf{A}$,
- $\mathcal{N}_r(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^m : \ \mathbf{x}\mathbf{A} = 0\}$, the left null space of $\mathbf{A}$.

Consider a matrix equation

$$\mathbf{AXB} = \mathbf{D}, \tag{4.1}$$

where $\mathbf{A} \in \mathbb{H}^{n\times n}$, $\mathbf{B} \in \mathbb{H}^{m\times m}$, $\mathbf{D} \in \mathbb{H}^{n\times m}$ are given, and $\mathbf{X} \in \mathbb{H}^{n\times m}$ is unknown. Let $Ind\,\mathbf{A} = k_1$ and $Ind\,\mathbf{B} = k_2$.

It's well known (see, e.g., [12]) that the equation (4.1) with restrictions

$$\mathcal{R}_r(\mathbf{X}) \subset \mathcal{R}_r(\mathbf{A}^{k_1}),\ \mathcal{N}_r(\mathbf{X}) \supset \mathcal{N}_r(\mathbf{B}^{k_2}),$$
$$\mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l(\mathbf{A}^{k_1}),\ \mathcal{N}_l(\mathbf{X}) \supset \mathcal{N}_l(\mathbf{B}^{k_2}),$$

has a unique solution $\mathbf{X} = \mathbf{A}^D\mathbf{D}\mathbf{B}^D$.

## 4.1. The Case of Hermitian Matrices

Denote $\mathbf{A}^{k_1}\mathbf{D}\mathbf{B}^{k_2} =: \widetilde{\mathbf{D}} = (\widetilde{d}_{ij}) \in \mathbb{H}^{n\times m}$.

**Theorem 4.1.** *If* $\mathbf{A}$, $\mathbf{B}$ *are Hermitian,* $\operatorname{rank}\mathbf{A}^{k_1+1} = \operatorname{rank}\mathbf{A}^{k_1} = r_1 \leq n$ *for* $\mathbf{A} \in \mathbb{H}^{n\times n}$*, and* $\operatorname{rank}\mathbf{B}^{k_2+1} = \operatorname{rank}\mathbf{B}^{k_2} = r_2 \leq m$ *for* $\mathbf{B} \in \mathbb{H}^{m\times m}$*, then, for the Drazin inverse solution* $\mathbf{X} = \mathbf{A}^D\mathbf{D}\mathbf{B}^D = (x_{ij}) \in \mathbb{H}^{n\times m}$ *of (4.1), we have*

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}\left(\mathbf{d}^{\,\mathbf{B}}_{\,.j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r_1,n}}\left|\left(\mathbf{A}^{k_1+1}\right)^{\beta}_{\beta}\right| \sum\limits_{\alpha\in I_{r_2,m}}\left|\left(\mathbf{B}^{k_2+1}\right)^{\alpha}_{\alpha}\right|}, \tag{4.2}$$

*or*

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{r_2,m}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{B}^{k_2+1}\right)_{j\,.}\left(\mathbf{d}^{\,\mathbf{A}}_{\,i\,.}\right)\right)^{\,\alpha}_{\,\alpha}}{\sum\limits_{\beta\in J_{r_1,n}}\left|\left(\mathbf{A}^{k_1+1}\right)^{\beta}_{\beta}\right| \sum\limits_{\alpha\in I_{r_2,m}}\left|\left(\mathbf{B}^{k_2+1}\right)^{\alpha}_{\alpha}\right|}, \tag{4.3}$$

*where*

$$\mathbf{d}^{\mathbf{B}}_{.\,j} = \left(\sum_{\alpha\in I_{r_2,m}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{B}^{k_2+1}\right)_{j.}\left(\tilde{\mathbf{d}}_{l.}\right)\right)^{\alpha}_{\alpha}\right) \in \mathbb{H}^{n\times 1}, \quad l = \overline{1,n}, \tag{4.4}$$

$$\mathbf{d}^{\mathbf{A}}_{i\,.} = \left(\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.i}\left(\tilde{\mathbf{d}}_{.t}\right)\right)^{\beta}_{\beta}\right) \in \mathbb{H}^{1\times m}, \quad t = \overline{1,m}, \tag{4.5}$$

*are the column vector and the row vector, respectively.* $\tilde{\mathbf{d}}_{i.}$ *and* $\tilde{\mathbf{d}}_{.j}$ *are the* $i$*-th row and the* $j$*-th column of* $\widetilde{\mathbf{D}}$ *for all* $i = \overline{1,n}$*,* $j = \overline{1,m}$*.*

*Proof.* An entry of the Drazin inverse solution $\mathbf{X} = \mathbf{A}^D\mathbf{D}\mathbf{B}^D = (x_{ij}) \in \mathbb{H}^{n\times m}$ is

$$x_{ij} = \sum_{s=1}^{m}\left(\sum_{t=1}^{n} a_{it}^D d_{ts}\right) b_{sj}^D \tag{4.6}$$

for all $i = \overline{1,n}$, $j = \overline{1,m}$, where by Theorem 3.6

$$a_{it}^D = \frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}\left(\mathbf{a}_{.t}^{(k_1)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|\left(\mathbf{A}^{k_1+1}\right)_{\beta}^{\beta}\right|}, \tag{4.7}$$

$$b_{sj}^D = \frac{\sum\limits_{\alpha\in I_{r_2,m}\{j\}} \operatorname{rdet}_j\left((\mathbf{B}^{k_2+1})_{j\,.}(\mathbf{b}_{s.}^{(k_2)})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r_2,m}} \left|(\mathbf{B}^{k_2+1})_{\alpha}^{\alpha}\right|}. \tag{4.8}$$

Denote by $\hat{\mathbf{d}}_{.s}$ the $s$-th column of $\mathbf{A}^{k_1}\mathbf{D} =: \hat{\mathbf{D}} = (\hat{d}_{ij}) \in \mathbb{H}^{n\times m}$ for all $s = \overline{1,m}$. It follows from $\sum\limits_t \mathbf{a}_{.\,t}^{(k_1)} d_{ts} = \hat{\mathbf{d}}_{.\,s}$ that

$$\sum_{t=1}^{n} a_{it}^D d_{ts} = \sum_{t=1}^{n} \frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}\left(\mathbf{a}_{.t}^{(k_1)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|\left(\mathbf{A}^{k_1+1}\right)_{\beta}^{\beta}\right|} \cdot d_{ts} =$$

$$\frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \sum\limits_{t=1}^{n} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}\left(\mathbf{a}_{.t}^{(k_1)}\right)\right)_{\beta}^{\beta} \cdot d_{ts}}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|\left(\mathbf{A}^{k_1+1}\right)_{\beta}^{\beta}\right|} = \frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}\left(\hat{\mathbf{d}}_{.\,s}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|\left(\mathbf{A}^{k_1+1}\right)_{\beta}^{\beta}\right|}$$

Suppose $\mathbf{e}_{s.}$ and $\mathbf{e}_{.\,s}$ are respectively the unit row-vector and the unit column-vector whose components are $0$, except the $s$-th components, which are $1$. Substituting (4.7) and (4.8) in (4.6), we obtain

$$x_{ij} = \sum_{s=1}^{m} \frac{\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}\left(\hat{\mathbf{d}}_{.\,s}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|\left(\mathbf{A}^{k_1+1}\right)_{\beta}^{\beta}\right|} \frac{\sum\limits_{\alpha\in I_{r_2,m}\{j\}} \operatorname{rdet}_j\left((\mathbf{B}^{k_2+1})_{j\,.}(\mathbf{b}_{s.}^{(k_2)})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r_2,m}} \left|(\mathbf{B}^{k_2+1})_{\alpha}^{\alpha}\right|}.$$

Since

$$\hat{\mathbf{d}}_{.\,s} = \sum_{t=1}^{n} \mathbf{e}_{.\,t}\hat{d}_{ts},\ \mathbf{b}_{s.}^{(k_2)} = \sum_{l=1}^{m} b_{sl}^{(k_2)}\mathbf{e}_{l.},\ \sum_{s=1}^{m} \hat{d}_{ts} b_{sl}^{(k_2)} = \widetilde{d}_{tl},$$

then we have

$$x_{ij} = \frac{\sum\limits_{s=1}^{m}\sum\limits_{l=1}^{m}\sum\limits_{t=1}^{n}\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}(\mathbf{e}_{.\,t})\right)_{\beta}^{\beta} \hat{d_{ts}} b_{sl}^{(k_2)} \sum\limits_{\alpha\in I_{r_2,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{B}^{k_2+1})_{j\,.}(\mathbf{e}_{l.})\right)_{\alpha}^{\alpha}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{k_1+1}\right)_{\beta}^{\beta}\right|\sum\limits_{\alpha\in I_{r_2,m}}\left|\left(\mathbf{B}^{k_2+1}\right)_{\alpha}^{\alpha}\right|} =$$

$$\frac{\sum\limits_{t=1}^{n}\sum\limits_{l=1}^{m}\sum\limits_{\beta\in J_{r_1,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}(\mathbf{e}_{.\,l})\right)_{\beta}^{\beta} \widetilde{d}_{tl} \sum\limits_{\alpha\in I_{r_2,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{B}^{k_2+1})_{j\,.}(\mathbf{e}_{t.})\right)_{\alpha}^{\alpha}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{k_1+1}\right)_{\beta}^{\beta}\right|\sum\limits_{\alpha\in I_{r_2,m}}\left|\left(\mathbf{B}^{k_2+1}\right)_{\alpha}^{\alpha}\right|}. \tag{4.9}$$

Denote by

$$d_{il}^{\mathbf{A}} := \sum_{\beta\in J_{r_1,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}\left(\tilde{\mathbf{d}}_{.\,l}\right)\right)_{\beta}^{\beta} = \sum_{t=1}^{n}\sum_{\beta\in J_{r_1,\,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}(\mathbf{e}_{.\,t})\right)_{\beta}^{\beta}\widetilde{d}_{tl}$$

the $l$-th component of a row-vector $\mathbf{d}_{i\,.}^{\mathbf{A}} = (d_{i1}^{\mathbf{A}}, ..., d_{im}^{\mathbf{A}})$ for all $l = \overline{1,m}$. Substituting it in (4.9), we have

$$x_{ij} = \frac{\sum\limits_{l=1}^{m} d_{it}^{\mathbf{A}} \sum\limits_{\alpha\in I_{r_2,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{B}^{k_2+1})_{j\,.}(\mathbf{e}_{l.})\right)_{\alpha}^{\alpha}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{k_1+1}\right)_{\beta}^{\beta}\right|\sum\limits_{\alpha\in I_{r_2,m}}\left|\left(\mathbf{B}^{k_2+1}\right)_{\alpha}^{\alpha}\right|}.$$

Since $\sum\limits_{l=1}^{m} d_{il}^{\mathbf{A}}\mathbf{e}_{l.} = \mathbf{d}_{i\,.}^{\mathbf{A}}$, then it follows (4.3).

If we denote by

$$d_{tj}^{\mathbf{B}} := \sum_{l=1}^{m}\widetilde{d}_{tl}\sum_{\alpha\in I_{r_2,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{B}^{k_2+1})_{j\,.}(\mathbf{e}_{l.})\right)_{\alpha}^{\alpha} = \sum_{\alpha\in I_{r_2,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{B}^{k_2+1})_{j\,.}(\widetilde{\mathbf{d}}_{t.})\right)_{\alpha}^{\alpha},$$

the $t$-th component of a column-vector $\mathbf{d}_{.\,j}^{\mathbf{B}} = (d_{1j}^{\mathbf{B}}, ..., d_{nj}^{\mathbf{B}})^T$ for all $t = \overline{1,n}$ and substitute it in (4.9), we obtain

$$x_{ij} = \frac{\sum\limits_{t=1}^{n}\sum\limits_{\beta\in J_{r_1,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}^{k_1+1}\right)_{.\,i}(\mathbf{e}_{.\,t})\right)_{\beta}^{\beta} d_{tj}^{\mathbf{B}}}{\sum\limits_{\beta\in J_{r_1,\,n}}\left|\left(\mathbf{A}^{k_1+1}\right)_{\beta}^{\beta}\right|\sum\limits_{\alpha\in I_{r_2,m}}\left|\left(\mathbf{B}^{k_2+1}\right)_{\alpha}^{\alpha}\right|}.$$

Since $\sum\limits_{t=1}^{n} \mathbf{e}_{.t} d_{tj}^{\mathbf{B}} = \mathbf{d}_{.j}^{\mathbf{B}}$, then it follows (4.2). □

Consider a matrix equation

$$\mathbf{A}\mathbf{X} = \mathbf{D}, \tag{4.10}$$

where $\mathbf{A} \in \mathbb{H}^{n\times n}$, $\mathbf{D} \in \mathbb{H}^{n\times m}$ are given, $\mathbf{A}$ is Hermitian, and $\mathbf{X} \in \mathbb{H}^{n\times m}$ is unknown. Let $Ind\,\mathbf{A} = k$. We denote $\mathbf{A}^k\mathbf{D} =: \hat{\mathbf{D}} = (\hat{d}_{ij}) \in \mathbb{H}^{n\times m}$. Putting $\mathbf{B} = \mathbf{I}$ in (4.1) we evidently obtain the following corollary.

**Corollary 4.1.** *If* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^{k} = r \leq n$ *for Hermitian* $\mathbf{A} \in \mathbb{H}^{n\times n}$*, then for the Drazin inverse solution* $\mathbf{X} = \mathbf{A}^D\mathbf{D} = (x_{ij})$ *of (4.10), we have*

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i \left( \left( \mathbf{A}^{k+1} \right)_{.i} \left( \hat{\mathbf{d}}_{.j} \right) \right) {}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left| \left( \mathbf{A}^{k+1} \right) {}_{\beta}^{\beta} \right|}, \tag{4.11}$$

*where* $\hat{\mathbf{d}}_{.j}$ *is the* $j$*-th column of* $\hat{\mathbf{D}}$ *for* $j = \overline{1,m}$*.*

Consider a matrix equation

$$\mathbf{X}\mathbf{B} = \mathbf{D}, \tag{4.12}$$

where $\mathbf{B} \in \mathbb{H}^{m\times m}$, $\mathbf{D} \in \mathbb{H}^{n\times m}$ are given, $\mathbf{B}$ is Hermitian and $\mathbf{X} \in \mathbb{H}^{n\times m}$ is unknown. Let $Ind\,\mathbf{B} = k$ and denote $\mathbf{D}\mathbf{B}^k =: \check{\mathbf{D}} = (\check{d}_{ij}) \in \mathbb{H}^{n\times m}$. Putting $\mathbf{A} = \mathbf{I}$ in (4.1) we evidently obtain the following corollary.

**Corollary 4.2.** *If* $\operatorname{rank}\mathbf{B}^{k+1} = \operatorname{rank}\mathbf{B}^{k} = r \leq n$ *for Hermitian* $\mathbf{B} \in \mathbb{H}^{n\times n}$*, then for the Drazin inverse solution* $\mathbf{X} = \mathbf{D}\mathbf{B}^D =: (x_{ij})$ *of (4.12), we have for* $i = \overline{1,n}$*,* $j = \overline{1,m}$

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \operatorname{rdet}_j \left( \left( \mathbf{B}^{k+1} \right)_{j\,.} \left( \check{\mathbf{d}}_{i\,.} \right) \right) {}_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,m}} \left| \left( \mathbf{B}^{k+1} \right) {}_{\alpha}^{\alpha} \right|}. \tag{4.13}$$

*where* $\check{\mathbf{d}}_{i.}$ *is the* $i$*-th row of* $\check{\mathbf{D}}$ *for* $i = \overline{1,n}$*.*

## 4.2. The Case of Arbitrary Matrices

Using the determinantal representations (3.14) for arbitrary $\mathbf{A} \in \mathbb{H}^{n\times n}$ and (3.15) for arbitrary $\mathbf{B} \in \mathbb{H}^{m\times m}$, we obtain the following theorem and corollaries by analogy to Theorem 4.1, Corollaries 4.1 and 4.2, respectively.

Denote $(\mathbf{A}^{2k_1+1})^*\mathbf{A}^{k_1}\mathbf{D}\mathbf{B}^{k_2}(\mathbf{B}^{2k_2+1})^* =: \widetilde{\mathbf{D}} = (\widetilde{d}_{ij}) \in \mathbb{H}^{n\times m}$.

**Theorem 4.2.** *If* $\operatorname{rank}\mathbf{A}^{k_1+1} = \operatorname{rank}\mathbf{A}^{k_1} = r_1 \leq n$ *for* $\forall\mathbf{A} \in \mathbb{H}^{n\times n}$*, and* $\operatorname{rank}\mathbf{B}^{k_2+1} = \operatorname{rank}\mathbf{B}^{k_2} = r_2 \leq m$ *for* $\forall\mathbf{B} \in \mathbb{H}^{m\times m}$*, then for the Drazin inverse solution* $\mathbf{X} = \mathbf{A}^D\mathbf{D}\mathbf{B}^D = (x_{ij}) \in \mathbb{H}^{n\times m}$ *of (4.1), we have*

$$x_{ij} = \frac{\sum\limits_{t=1}^{n} a_{it}^{(k_1)} \sum\limits_{\beta\in J_{r_1,\,n}\{t\}} \operatorname{cdet}_t\left(\left(\mathbf{A}^{2k_1+1}\right)^*\left(\mathbf{A}^{2k_1+1}\right)_{.\,t}\left(\mathbf{d}^{\mathbf{B}}_{.\,j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|\left(\mathbf{A}^{2k_1+1}\right)^*\left(\mathbf{A}^{2k_1+1}\right)_{\beta}^{\beta}\right| \sum\limits_{\alpha\in I_{r_2,\,m}} \left|\left(\mathbf{B}^{2k_2+1}\left(\mathbf{B}^{2k_2+1}\right)^*\right)_{\alpha}^{\alpha}\right|} \tag{4.14}$$

*or*

$$x_{ij} = \frac{\sum\limits_{s=1}^{m} \left(\sum\limits_{\alpha\in I_{r_2,\,m}\{s\}} \operatorname{rdet}_s\left(\left(\mathbf{B}^{2k_2+1}\left(\mathbf{B}^{2k_2+1}\right)^*\right)_{.\,s}\left(\mathbf{d}^{\mathbf{A}}_{i\,.}\right)\right)_{\alpha}^{\alpha}\right) a_{sj}^{(k_2)}}{\sum\limits_{\beta\in J_{r_1,\,n}} \left|\left(\mathbf{A}^{2k_1+1}\right)^*\left(\mathbf{A}^{2k_1+1}\right)_{\beta}^{\beta}\right| \sum\limits_{\alpha\in I_{r_2,\,m}} \left|\left(\mathbf{B}^{2k_2+1}\left(\mathbf{B}^{2k_2+1}\right)^*\right)_{\alpha}^{\alpha}\right|}, \tag{4.15}$$

*where*

$$\mathbf{d}^{\mathbf{B}}_{.\,j} = \left(\sum_{s=1}^{m}\left(\sum_{\alpha\in I_{r_2,\,m}\{s\}} \operatorname{rdet}_s\left(\left(\mathbf{B}^{2k_2+1}\left(\mathbf{B}^{2k_2+1}\right)^*\right)_{.\,s}\left(\widetilde{\mathbf{d}}_{q.}\right)\right)_{\alpha}^{\alpha}\right) a_{sj}^{(k_2)}\right) \in \mathbb{H}^{n\times 1}, \tag{4.16}$$

*and*

$$\mathbf{d}^{\mathbf{A}}_{i\,.} = \left(\sum_{t=1}^{n} a_{it}^{(k_1)} \sum_{\beta\in J_{r_1,\,n}\{t\}} \operatorname{cdet}_t\left(\left(\mathbf{A}^{2k_1+1}\right)^*\left(\mathbf{A}^{2k_1+1}\right)_{.\,t}\left(\widetilde{\mathbf{d}}_{.p}\right)\right)_{\beta}^{\beta}\right) \in \mathbb{H}^{1\times m}, \tag{4.17}$$

*and* $\tilde{\mathbf{d}}_{p.}$*,* $\tilde{\mathbf{d}}_{.q}$ *are the* $p$*-th row and the* $q$*-th column of* $\widetilde{\mathbf{D}}$*, respectively, for all* $q = \overline{1,n}$*,* $p = \overline{1,m}$*.*

**Corollary 4.3.** *If* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^{k} = r \leq n$ *for* $\mathbf{A} \in \mathbb{H}^{n\times n}$*, then, for the*

*Drazin inverse solution* $\mathbf{X} = \mathbf{A}^D\mathbf{D} = (x_{ij})$ *of (4.10), we have*

$$x_{ij} = \frac{\sum\limits_{t=1}^{n} a_{it}^{(k)} \sum\limits_{\beta \in J_{r,\,n}\{t\}} \operatorname{cdet}_t \left( \left(\mathbf{A}^{2k+1}\right)^* \left(\mathbf{A}^{2k+1}\right)_{.\,t} \left(\hat{\mathbf{d}}_{\,.\,j}\right)\right) {}_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\,n}} \left| \left(\mathbf{A}^{2k+1}\right)^* \left(\mathbf{A}^{2k+1}\right) {}_{\beta}^{\beta} \right|}, \tag{4.18}$$

*where* $\hat{\mathbf{d}}_{.j}$ *is the* $j$*-th column of* $\hat{\mathbf{D}} = (\mathbf{A}^{2k_1+1})^* \mathbf{A}^{k_1} \mathbf{D}$ *for all* $j = \overline{1,m}$, $i = \overline{1,n}$.

**Corollary 4.4.** *If* $\operatorname{rank} \mathbf{B}^{k+1} = \operatorname{rank} \mathbf{B}^k = r \leq m$ *for* $\mathbf{B} \in \mathbb{H}^{m \times m}$*, then, for the Drazin inverse solution* $\mathbf{X} = \mathbf{D}\mathbf{B}^D =: (x_{ij})$ *of (4.12), we have*

$$x_{ij} = \frac{\sum\limits_{s=1}^{m} \left( \sum\limits_{\alpha \in I_{r,\,m}\{s\}} \operatorname{rdet}_s \left( \left( \mathbf{B}^{2k+1} \left(\mathbf{B}^{2k+1}\right)^* \right)_{.s} \left(\check{\mathbf{d}}_{\,i\,.}\right) \right) {}_{\alpha}^{\alpha} \right) a_{sj}^{(k)}}{\sum\limits_{\alpha \in I_{r,\,m}} \left| \left( \mathbf{B}^{2k+1} \left(\mathbf{B}^{2k+1}\right)^* \right) {}_{\alpha}^{\alpha} \right|}, \tag{4.19}$$

*where* $\check{\mathbf{d}}_{i.}$ *is the* $i$*-th row of* $\check{\mathbf{D}} = \mathbf{D}\mathbf{B}^{k_2} (\mathbf{B}^{2k_2+1})^*$ *for all* $i = \overline{1,n}$, $j = \overline{1,m}$.

### 4.3. Examples

In this section, we give examples to illustrate our results.

1. Let us consider the matrix equation

$$\mathbf{AXB} = \mathbf{D}, \tag{4.20}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & k & -i \\ -k & 2 & j \\ i & -j & 1 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 1 & i \\ k & 1 \\ 1 & j \end{pmatrix}.$$

Since $\mathbf{A}^2 = \begin{pmatrix} 3 & 4k & -3i \\ -4k & 6 & 4j \\ 3i & -4j & 3 \end{pmatrix}$, $\det \mathbf{A} = \det \mathbf{A}^2 = 0$, and $\det \begin{pmatrix} 1 & k \\ -k & 2 \end{pmatrix} = 1$, $\det \begin{pmatrix} 3 & 4k \\ -4k & 6 \end{pmatrix} = 2$, then, by Theorem 2.11, $Ind\,\mathbf{A} = 1$ and $r_1 = \operatorname{rank} \mathbf{A} = 2$. Similarly, since $\mathbf{B}^2 = \begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix}$, then $Ind\,\mathbf{B} = 1$ and $r_2 = \operatorname{rank} \mathbf{B} = 1$.

Because $\mathbf{A}$ and $\mathbf{B}$ are Hermitian, we shall find the Drazin inverse solution $\mathbf{X}^d = (x_{ij}^d)$ of (4.20) by the equations (4.2)-(4.4). We have $\sum\limits_{\alpha\in I_{1,\,2}} \left|\left(\mathbf{B}^2\right)_{\alpha}^{\alpha}\right| = 2+2=4$,

$$\sum_{\beta\in J_{2,\,3}} \left|\left(\mathbf{A}^2\right)_{\beta}^{\beta}\right| = \det\begin{pmatrix} 3 & 4k \\ -4k & 6 \end{pmatrix} + \det\begin{pmatrix} 3 & -3i \\ 3i & 3 \end{pmatrix} + \det\begin{pmatrix} 6 & 4j \\ -4j & 3 \end{pmatrix} = 4.$$

Since

$$\widetilde{\mathbf{D}} = \mathbf{ADB} = \begin{pmatrix} 1-i & 1+i \\ -i+j & 1-k \\ 1+i & -1+i \end{pmatrix},$$

then by (4.4)

$$\mathbf{d}_{.\,j}^{\mathbf{B}} = \left(\sum_{\alpha\in I_{1,2}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{B}^2\right)_{1.}\left(\tilde{\mathbf{d}}_{l.}\right)\right)_{\alpha}^{\alpha}\right) \in \mathbb{H}^{n\times 1}, \quad l=1,2,3; \;\; j=1,2.$$

Thus, we have

$$\mathbf{d}_{.1}^{\mathbf{B}} = \begin{pmatrix} 1-i \\ -i+j \\ 1+i \end{pmatrix}, \quad \mathbf{d}_{.2}^{\mathbf{B}} = \begin{pmatrix} 1+i \\ 1-k \\ -1+i \end{pmatrix}.$$

So

$$\left(\mathbf{A}^2\right)_{.\,1}\left(\mathbf{d}_{.1}^{\mathbf{B}}\right) = \begin{pmatrix} 1-i & 4k & -3i \\ -i+j & 6 & 4j \\ 1+i & -4j & 3 \end{pmatrix},$$

and finally we obtain

$$x_{11}^d = \frac{\sum\limits_{\beta\in J_{2,\,3}\{1\}} \operatorname{cdet}\left(\left(\mathbf{A}^2\right)_{.\,1}\left(\mathbf{d}_{.1}^{\mathbf{B}}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{2,3}} \left|(\mathbf{A}^2)_{\beta}^{\beta}\right| \sum\limits_{\alpha\in I_{1,2}} |(\mathbf{B}^2)_{\alpha}^{\alpha}|} =$$

$$\frac{1}{16}\left(\operatorname{cdet}_1\begin{pmatrix} 1-i & 4k \\ -i+j & 6 \end{pmatrix} + \operatorname{cdet}_1\begin{pmatrix} 1-i & -3i \\ 1+i & 3 \end{pmatrix}\right) = \frac{3-i+2j}{8}.$$

Similarly,

$$x_{12}^d = \frac{1}{16}\left(\mathrm{cdet}_1\begin{pmatrix}1+i & 4k\\ 1-k & 6\end{pmatrix} + \mathrm{cdet}_1\begin{pmatrix}1+i & -3i\\ 1+i & 3\end{pmatrix}\right) = \frac{1+3i-2k}{8},$$

$$x_{21}^d = \frac{1}{16}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1-i\\ -4k & -i+j\end{pmatrix} + \mathrm{cdet}_1\begin{pmatrix}-i+j & 4j\\ 1+i & 3\end{pmatrix}\right) = \frac{-3i-j+4k}{8},$$

$$x_{22}^d = \frac{1}{16}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1+i\\ -4k & 1-k\end{pmatrix} + \mathrm{cdet}_1\begin{pmatrix}1-k & 4j\\ -1+i & 3\end{pmatrix}\right) = \frac{3+4j+k}{8},$$

$$x_{31}^d = \frac{1}{16}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1-i\\ 3i & 1+i\end{pmatrix} + \mathrm{cdet}_2\begin{pmatrix}6 & -i+j\\ -4j & 1+i\end{pmatrix}\right) = \frac{1+3i+2k}{8},$$

$$x_{32}^d = \frac{1}{16}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1+i\\ 3i & -1+i\end{pmatrix} + \mathrm{cdet}_2\begin{pmatrix}6 & 1-k\\ -4j & -1+i\end{pmatrix}\right) = \frac{-3+i+2j}{8}.$$

So,

$$\mathbf{X}^d = \frac{1}{8}\begin{pmatrix}3-i+2j & 1+3i-2k\\ -3i-j+4k & 3+4j+k\\ 1+3i+2k & -3+i+2j\end{pmatrix}$$

is the Drazin inverse solution of (4.20).

2. Let us consider the matrix equation

$$\mathbf{AX} = \mathbf{D}, \tag{4.21}$$

where

$$\mathbf{A} = \begin{pmatrix}i & j & k\\ 1 & -k & j\\ 1 & 0 & i\end{pmatrix},\ \mathbf{D} = \begin{pmatrix}1 & i\\ k & 1\\ 1 & j\end{pmatrix}.$$

Since $\mathbf{A}^2 = \begin{pmatrix}-1+j+k & -i+k & -1\\ i+j-k & -1+j & i\\ 2i & j & -1+k\end{pmatrix}$, $\mathbf{A}^*\mathbf{A} = \begin{pmatrix}3 & -2k & i+2j\\ 2k & 2 & -2i\\ -i+2j & 2i & 3\end{pmatrix}$,

$$(\mathbf{A}^2)^*\mathbf{A}^2 = \begin{pmatrix}10 & 2+2j-6k & 2+2i+4j+2k\\ 2-2j+6k & 5 & -3i+j+2k\\ 2-2i-4j-2k & 3-j-2k & 4\end{pmatrix},$$

$$\det \mathbf{A}^*\mathbf{A} = \det(\mathbf{A}^2)^*\mathbf{A}^2 = 0, \det \begin{pmatrix} 3 & -2k \\ 2k & 2 \end{pmatrix} = 2,$$

$$\det \begin{pmatrix} 10 & 2+2j-6k \\ 2-2j+6k & 5 \end{pmatrix} = 6,$$

then, by Theorem 2.11, $Ind\, \mathbf{A} = 1$ and $r = \operatorname{rank} \mathbf{A} = 2$. We shall find the Drazin inverse solution $\mathbf{X}^d = (x_{ij}^d)$ of (4.21) by (4.18). Since

$$(\mathbf{A}^3)^*\mathbf{A}^3 = \begin{pmatrix} 23 & 2+3i+5j-17k & 8+4i+15j+2k \\ 2-3i-5j+17k & 15 & 3-13i+2j+5k \\ 8-4i-15j-2k & 3+13i-2j-5k & 15 \end{pmatrix},$$

then

$$\sum_{\beta \in J_{2,\,3}} \left| \left( (\mathbf{A}^3)^*\mathbf{A}^3 \right) {}_{\beta}^{\beta} \right| = \det \begin{pmatrix} 23 & 2+3i+5j-17k \\ 2-3i-5j+17k & 15 \end{pmatrix}$$

$$+ \det \begin{pmatrix} 15 & 3-13i+2j+5k \\ 3+13i-2j-5k & 15 \end{pmatrix}$$

$$+ \det \begin{pmatrix} 23 & 8+4i+15j+2k \\ 8-4i-15j-2k & 15 \end{pmatrix} = 72.$$

Further,

$$\hat{\mathbf{D}} = (\mathbf{A}^3)^*\mathbf{A}\mathbf{D} = \begin{pmatrix} -11-9i-6j+2k & 9-6i-j \\ -5+5i-4j-10k & -1-2i-7j+6k \\ -10-4i+7j-3k & 3-4i-7j-4k \end{pmatrix},$$

and

$$\left( (\mathbf{A}^3)^*\mathbf{A}^3 \right)_{.\,1} \left( \hat{\mathbf{d}}_{.\,1} \right) = \begin{pmatrix} -11-9i-6j+2k & 2+3i+5j-17k & 8+4i+15j+2k \\ -5+5i-4j-10k & 15 & 3-13i+2j+5k \\ -10-4i+7j-3k & 3+13i-2j-5k & 15 \end{pmatrix}.$$

Therefore, finally we obtain

$$x_{11}^{d} = \frac{\sum_{t=1}^{3} a_{1t} \sum_{\beta \in J_{2,3}\{t\}} \mathrm{cdet}_t \left( \left( \left(\mathbf{A}^3\right)^* \mathbf{A}^3 \right)_{.\,t} \left( \hat{\mathbf{d}}_{\,.\,1} \right) \right) {}_{\beta}^{\beta}}{\sum_{\beta \in J_{2,3}} \left| \left( (\mathbf{A}^3)^* \mathbf{A}^3 \right) {}_{\beta}^{\beta} \right|} =$$

$$\frac{i}{76} \left( \mathrm{cdet}_1 \begin{pmatrix} 1 & 2+3i+5j-17k \\ k & 5 \end{pmatrix} + \mathrm{cdet}_1 \begin{pmatrix} 1 & 8+4i+15j+2k \\ 1 & 15 \end{pmatrix} \right) +$$

$$\frac{j}{76} \left( \mathrm{cdet}_2 \begin{pmatrix} 23 & 1 \\ 2-3i-5j+17k & k \end{pmatrix} + \mathrm{cdet}_1 \begin{pmatrix} k & 3-13i+2j+5k \\ 1 & 15 \end{pmatrix} \right) +$$

$$\frac{k}{76} \left( \mathrm{cdet}_2 \begin{pmatrix} 23 & 1 \\ 8-4i-15j-2k & 1 \end{pmatrix} + \mathrm{cdet}_2 \begin{pmatrix} 15 & k \\ 3+13i-2j-5k & 1 \end{pmatrix} \right) =$$

$$\frac{1}{76}(7-17i+5j-3k)$$

Similarly,

$$x_{12}^{d} = \frac{i}{76}(13+29i-13j+13k) + \frac{j}{76}(37+3i+14j+18k) - \frac{k}{76}(7+21i-42j+10k) =$$
$$\frac{1}{76}(-33-11i+3j-23k),$$

$$x_{21}^{d} = \frac{1}{76}(-5-9i-12j-4k) + \frac{k}{76}(-5+18i-5j+8k) + \frac{j}{76}(25+6i+28j-k) =$$
$$\frac{1}{76}(-49-13i+29j-15k),$$

$$x_{22}^{d} = \frac{1}{76}(13+29i-13j+13k) + \frac{k}{76}(37+3i+14j+18k) - \frac{j}{76}(7+21i-42j+10k) =$$
$$\frac{1}{76}(-47+5i-17j+71k),$$

$$x_{31}^{d} = \frac{1}{76}(-5-9i-12j-4k) + \frac{0}{76}(-5+18i-5j+8k) + \frac{i}{76}(25+6i+28j-k) =$$
$$\frac{1}{76}(-11+16i-11j+24k),$$

$$x_{32}^{d} = \frac{1}{76}(13+29i-13j+13k) + \frac{0}{76}(37+3i+14j+18k) - \frac{i}{76}(7+21i-42j+10k) =$$
$$\frac{1}{76}(34+22i-3j+55k).$$

Thus,we have the Drazin inverse solution of (4.21),

$$\mathbf{X}^d = \frac{1}{76} \begin{pmatrix} 7-17i+5j-3k & -33-11i+3j-23k \\ -49-13i+29j-15k & -47+5i-17j+71k \\ -11+16i-11j+24k & 34+22i-3j+55k \end{pmatrix}.$$

# 5. Applications of the Determinantal Representations of the Drazin Inverse to Some Differential Matrix Equations

In [40], applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients have been done. In [41], we recently have obtained determinantal representations of solutions of some singular differential complex-valued matrix equations. In this chapter we extend studies conducted in [41] from the complex field to the quaternion skew field.

## 5.1. Background for Quaternion-valued Differential Equations (QDE)

Consider a quaternion-valued function of real variable, $\mathbf{f}:\mathbb{R}\to\mathbb{H}$, ($t\in\mathbb{R}$ is a real variable), such that $\mathbf{f}(t)=f_0(t)+f_1(t)\mathbf{i}+f_2(t)\mathbf{j}+f_3(t)\mathbf{k}$. The first derivative of a quaternionic function $\mathbf{f}(t)$ with respect to the real variable $t$ denote by,

$$\mathbf{f}'(t):=\frac{\mathrm{d}\mathbf{f}(t)}{\mathrm{d}t}=\frac{\mathrm{d}f_0(t)}{\mathrm{d}t}+\frac{\mathrm{d}f_1(t)}{\mathrm{d}t}\mathbf{i}+\frac{\mathrm{d}f_2(t)}{\mathrm{d}t}\mathbf{j}+\frac{\mathrm{d}f_3(t)}{\mathrm{d}t}\mathbf{k}.$$

It is easy to prove the following proposition on properties of the derivative of quaternionic functions.

**Proposition 5.1.** *If* $\mathbf{q}:\mathbb{R}\to\mathbb{H}$ *and* $\mathbf{r}:\mathbb{R}\to\mathbb{H}$ *are differentiable, then* $(\mathbf{q}\pm\mathbf{r})(t)$, $\mathbf{qr}(t)$ *and, for any integer* $n\geq 1$, $\mathbf{q}^n$ *are differentiable, and*

$$(\mathbf{q}\pm\mathbf{r})'(t)=\mathbf{q}'(t)\pm\mathbf{r}'(t), \tag{5.1}$$

$$(\mathbf{qr})'(t)=\mathbf{q}'(t)\mathbf{r}(t)+\mathbf{q}(t)\mathbf{r}'(t), \tag{5.2}$$

$$[\mathbf{q}^n(t)]'=\sum_{j=0}^{n-1}\mathbf{q}^j(t)\mathbf{q}'(t)\mathbf{q}^{n-j}(t). \tag{5.3}$$

If $f_i(t)$ for all $l=\overline{0,3}$ is integrable on $[a,b]\subset\mathbb{R}$, the $\mathbf{f}(t)$ is integrable and

$$\int_a^b\mathbf{f}(t)\mathrm{d}t=\int_a^b f_0(t)\mathrm{d}t+\int_a^b f_1(t)\mathrm{d}t\,\mathbf{i}+\int_a^b f_2(t)\mathrm{d}t\,\mathbf{j}+\int_a^b f_3(t)\mathrm{d}t\,\mathbf{k}.$$

Consider a matrix valued function $\mathbf{A}(t) = (\mathbf{a}_{ij}(t)) \in \mathbb{H}^{n\times n} \otimes \mathbb{R}$, where $\mathbf{a}_{ij}(t)$ are quaternion-valued functions with the real variable $t$ for all $i, j = \overline{1, n}$. Then

$$\frac{\mathrm{d}\mathbf{A}(t)}{\mathrm{d}t} = \left(\frac{\mathrm{d}\mathbf{a}_{ij}(t)}{\mathrm{d}t}\right)_{n\times n}, \qquad \int_a^b \mathbf{A}(t)\mathrm{d}t = \left(\int_a^b \mathbf{a}_{ij}(t)\mathrm{d}t\right)_{n\times n}.$$

We need the exponential of $q \in \mathbb{H}$ that can be defined by putting,

$$\exp q = \sum_{n=0}^{\propto} \frac{q^n}{n!}. \tag{5.4}$$

From the definition of a quaternionic exponential (5.4), we evidently have the following properties.

**Proposition 5.2.** *If* $q, r \in \mathbb{H}$ *are such that* $qr = rq$, *then* $\exp(q + r) = (\exp q)(\exp r)$.

**Proposition 5.3.** *If* $\mathbf{q} : \mathbb{R} \to \mathbb{H}$ *is differentiable and* $\mathbf{q}'(t)\mathbf{q}(t) = \mathbf{q}(t)\mathbf{q}'(t)$, *then*

$$[\exp \mathbf{q}(t)]' = [\exp \mathbf{q}(t)]\, \mathbf{q}'(t).$$

In [42], the linear quaternion differential equations,

$$\mathbf{q}'(t) = \mathbf{a}(t)\mathbf{q}(t), \tag{5.5}$$

and

$$\mathbf{q}'(t) = \mathbf{q}(t)\mathbf{a}(t), \tag{5.6}$$

with the initial condition $q(t_0) = q_0$ have been considered and the following proposition has been derived.

**Proposition 5.4.** *Let* $\mathbf{q}(t) = \mathbf{\Phi}_l(t)q_0$ *and* $\mathbf{q}(t) = q_0\mathbf{\Phi}_r(t)$ *be solutions of (5.5) and (5.6), respectively. If*

$$\mathbf{a}(t)\int_{t_0}^{t} \mathbf{a}(\tau)\mathrm{d}\tau = \int_{t_0}^{t} \mathbf{a}(\tau)\mathrm{d}\tau\, \mathbf{a}(t), \tag{5.7}$$

*then*

$$\mathbf{\Phi}_l(t) = \mathbf{\Phi}_r(t) = \exp\left(\int_{t_0}^{t} \mathbf{a}(\tau)\mathrm{d}t\right).$$

If $\mathbf{a}$ is constant, then, evidently, $\int_{t_0}^{t} \mathbf{a}(\tau)\mathrm{d}t = \mathbf{a}(t-t_0)$, and $\mathbf{\Phi}_l(t) = \mathbf{\Phi}_r(t) = \exp(\mathbf{a}(t-t_0))$.

The similar result has been obtained in [43] as well. In [43], the following nonhomogeneous differential equation corresponding to (5.5) has been considered,

$$\mathbf{q}'(t) = \mathbf{a}(t)\mathbf{q}(t) + \mathbf{f}(t), \tag{5.8}$$

where $\mathbf{f} : [0,T] \to \mathbb{H}$ and $\mathbf{a} : [0,T] \to \mathbb{H}$. It has been shown, if condition (5.7) is satisfied, then the solutions of (5.8) are given by

$$\mathbf{q}(t) = \exp\left(\int_0^t \mathbf{a}(\tau)\mathrm{d}\tau\right)\left(\mathbf{q}(0) + \int_0^t \exp\left(\int_0^s (-\mathbf{a}(\tau))\,\mathrm{d}\tau\right)\mathbf{f}(s)\mathrm{d}s\right), \quad (t \in [0,T]). \tag{5.9}$$

In the special case when $\mathbf{a}$ is constant and $\mathbf{q}(0) = 1$, then the solutions of (5.8) are given by

$$\mathbf{q}(t) = \exp(\mathbf{a}t)\left(\int_0^t \exp(-\mathbf{a}s)\,\mathbf{f}(s)\mathrm{d}s\right), \quad (t \in [0,T]). \tag{5.10}$$

## 5.2. Determinantal Representations of Solutions of Some Singular Differential Quaternion-Matrix Equations

Consider the matrix differential equation

$$\mathbf{X}' + \mathbf{A}\mathbf{X} = \mathbf{B}, \tag{5.11}$$

where $\mathbf{A} \in \mathbb{H}^{n\times n}$, $\mathbf{B} \in \mathbb{H}^{n\times n}$ are given, $\mathbf{X} \in \mathbb{H}^{n\times n}$ is unknown. By (5.10) the general solution of (5.11) is found to be

$$\mathbf{X}(t) = \exp(-\mathbf{A}t)\left(\int \exp(\mathbf{A}t)\,dt\right)\mathbf{B}.$$

If $\mathbf{A}$ is invertible, then

$$\int \exp(\mathbf{A}t)\,dt = \mathbf{A}^{-1}\exp(\mathbf{A}t) + \mathbf{G}, \tag{5.12}$$

where $\mathbf{G}$ is an arbitrary $n \times n$ quaternionic matrix.

Since $\mathbf{A}^{-1}\exp(\mathbf{A}) = \exp(\mathbf{A})\mathbf{A}^{-1}$, then the general solution of (5.11) is $\mathbf{X}(t) = \{\mathbf{A}^{-1} + \exp(-\mathbf{A}t)\,\mathbf{G}\}\mathbf{B}$. If $\mathbf{A}$ is noninvertible, then due to [30] the following theorem can be expended to quaternion matrices.

**Theorem 5.1.** *If* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *has index* $k$*, then*

$$\int \exp(\mathbf{A}t)\,\mathrm{d}t = \mathbf{A}^D \exp(\mathbf{A}t) + (\mathbf{I} - \mathbf{A}\mathbf{A}^D)t\left[\mathbf{I} + \frac{\mathbf{A}}{2}t + \frac{\mathbf{A}^2}{3!}t^2 + ... + \frac{\mathbf{A}^{k-1}}{k!}t^{k-1}\right] + \mathbf{G}. \tag{5.13}$$

*Proof.* Differentiate the right-hand side of (5.13), and use the series expansion for $\exp(\mathbf{A}t)$. □

Using (5.13) and the series expansion for $\exp(-\mathbf{A}t)$, we get an explicit form for a general solution of (5.11),

$$\mathbf{X}(t) = \left\{\mathbf{A}^D + (\mathbf{I} - \mathbf{A}\mathbf{A}^D)t\left(\mathbf{I} - \frac{\mathbf{A}}{2}t + \frac{\mathbf{A}^2}{3!}t^2 - ...(-1)^{k-1}\frac{\mathbf{A}^{k-1}}{k!}t^{k-1}\right) + \mathbf{G}\right\}\mathbf{B}.$$

If we put $\mathbf{G} = \mathbf{0}$, then the following partial solution of (5.11) is obtained,

$$\mathbf{X}(t) = \mathbf{A}^D\mathbf{B} + (\mathbf{B} - \mathbf{A}^D\mathbf{A}\mathbf{B})t - \frac{1}{2}(\mathbf{A}\mathbf{B} - \mathbf{A}^D\mathbf{A}^2\mathbf{B})t^2 + ...\frac{(-1)^{k-1}}{k!}(\mathbf{A}^{k-1}\mathbf{B} - \mathbf{A}^D\mathbf{A}^k\mathbf{B})t^k. \tag{5.14}$$

**Theorem 5.2.** *If* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *has index* $k$ *and* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^k = r \le n$*, then the partial solution (5.14),* $\mathbf{X}(t) = (x_{ij})$*, possess the following determinantal representation,*

*1. when* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *is Hermitian,*

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left(\mathbf{A}^{k+1}_{.\,i}\left(\widehat{\mathbf{b}}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{A}^{k+1}\right)^{\beta}_{\beta}\right|} + \left(b_{ij} - \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left(\mathbf{A}^{k+1}_{.\,i}\left(\widehat{\mathbf{b}}^{(k+1)}_{.j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{A}^{k+1}\right)^{\beta}_{\beta}\right|}\right)t-$$

$$-\frac{1}{2}\left(\widehat{b}^{(1)}_{ij} - \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left(\mathbf{A}^{k+1}_{.\,i}\left(\widehat{\mathbf{b}}^{(k+2)}_{.j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{A}^{k+1}\right)^{\beta}_{\beta}\right|}\right)t^2 + ...$$

$$\frac{(-1)^k}{k!}\left(\widehat{b}^{(k-1)}_{ij} - \frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}} \operatorname{cdet}_i\left(\mathbf{A}^{k+1}_{.\,i}\left(\widehat{\mathbf{b}}^{(2k)}_{.j}\right)\right)^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{A}^{k+1}\right)^{\beta}_{\beta}\right|}\right)t^k \tag{5.15}$$

*where* $\mathbf{A}^l\mathbf{B} =: \widehat{\mathbf{B}}^{(l)} = \left(\widehat{b}^{(l)}_{ij}\right) \in \mathbb{H}^{n\times n}$ *for all* $l = \overline{k, 2k}$*;*

*2. when* $\mathbf{A}$ *is arbitrary,*

$$x_{ij}=\frac{\sum\limits_{s=1}^{n}a_{it}^{(k)}\sum\limits_{\beta\in J_{r,\,n}\{s\}}\operatorname{cdet}_s\left(\left(\mathbf{A}^{2k+1}\right)^*\left(\mathbf{A}^{2k+1}\right)_{.\,s}\left(\widehat{\mathbf{d}}_{.j}^{(0)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^{2k+1}\right)^*\left(\mathbf{A}^{2k+1}\right)_{\beta}^{\beta}\right|}$$

$$+\left(b_{ij}-\frac{\sum\limits_{s=1}^{n}a_{is}^{(k)}\sum\limits_{\beta\in J_{r,\,n}\{s\}}\operatorname{cdet}_s\left(\left(\mathbf{A}^{2k+1}\right)^*\left(\mathbf{A}^{2k+1}\right)_{.\,s}\left(\widehat{\mathbf{d}}_{.j}^{(1)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^{2k+1}\right)^*\left(\mathbf{A}^{2k+1}\right)_{\beta}^{\beta}\right|}\right)t$$

$$-\frac{1}{2}\left(\widehat{b}_{ij}^{(1)}-\frac{\sum\limits_{t=1}^{n}a_{it}^{(k)}\sum\limits_{\beta\in J_{r,\,n}\{s\}}\operatorname{cdet}_s\left(\left(\mathbf{A}^{2k+1}\right)^*\left(\mathbf{A}^{2k+1}\right)_{.\,s}\left(\widehat{\mathbf{d}}_{.j}^{(2)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^{2k+1}\right)^*\left(\mathbf{A}^{2k+1}\right)_{\beta}^{\beta}\right|}\right)t^2+\ldots$$

$$\frac{(-1)^k}{k!}\left(\widehat{b}_{ij}^{(k-1)}-\frac{\sum\limits_{s=1}^{n}a_{is}^{(k)}\sum\limits_{\beta\in J_{r,\,n}\{s\}}\operatorname{cdet}_s\left(\left(\mathbf{A}^{2k+1}\right)^*\left(\mathbf{A}^{2k+1}\right)_{.\,s}\left(\widehat{\mathbf{d}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^{2k+1}\right)^*\left(\mathbf{A}^{2k+1}\right)_{\beta}^{\beta}\right|}\right)t^k \tag{5.16}$$

*where* $(\mathbf{A}^{2k+1})^*\mathbf{A}^{k+l}\mathbf{B}=\hat{\mathbf{A}}\mathbf{A}^l\mathbf{B}=:\widehat{\mathbf{D}}^{(l)}=(\widehat{d}_{ij}^{(l)})\in\mathbb{H}^{n\times n}$ *for all* $l=\overline{1,k}$ *and for all* $i,j=\overline{1,n}$*.*

*Proof.* 1. Using the determinantal representation of $\mathbf{A}^D$ by (3.6), we obtain the following determinantal representation of the matrix $\mathbf{A}^D\mathbf{A}^m\mathbf{B}:=(y_{ij})$,

$$y_{ij}=\sum_{s=1}^{n}a_{is}^{D}\sum_{t=1}^{n}a_{st}^{(m)}b_{tj}=\sum_{\beta\in J_{r,n}\{i\}}\frac{\sum\limits_{s=1}^{n}\operatorname{cdet}_i\left(\mathbf{A}_{.\,i}^{k+1}\left(\mathbf{a}_{.s}{}^{(k)}\right)\right)_{\beta}^{\beta}\cdot\sum\limits_{t=1}^{n}a_{st}^{(m)}b_{tj}}{\sum\limits_{\beta\in J_{r,n}}\left|(\mathbf{A}^{k+1})_{\beta}^{\beta}\right|}=$$

$$\sum_{\beta\in J_{r,n}\{i\}}\frac{\sum\limits_{t=1}^{n}\operatorname{cdet}_i\left(\mathbf{A}_{.\,i}^{k+1}\left(\mathbf{a}_{.t}{}^{(k+m)}\right)\right)_{\beta}^{\beta}\cdot b_{tj}}{\sum\limits_{\beta\in J_{r,n}}\left|(\mathbf{A}^{k+1})_{\beta}^{\beta}\right|}=\frac{\sum\limits_{\beta\in J_{r,\,n}\{i\}}\operatorname{cdet}_i\left(\mathbf{A}_{.\,i}^{k+1}\left(\widehat{\mathbf{b}}_{.j}^{(k+m)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\mathbf{A}^{k+1}\right)_{\beta}^{\beta}\right|}$$

for all $i,j=\overline{1,n}$ and $m=\overline{1,k}$. From this and the determinantal representation of the Drazin inverse solution (4.11), it follows (5.15).

2. The proof of (5.16) is similar to the proof of (5.15) by using the determinantal representation of $\mathbf{A}^D$ by (3.14). □

Consider the matrix differential equation

$$\mathbf{X}' + \mathbf{X}\mathbf{A} = \mathbf{B} \tag{5.17}$$

where $\mathbf{A} \in \mathbb{H}^{n\times n}$, $\mathbf{B} \in \mathbb{H}^{n\times n}$ are given, $\mathbf{X} \in \mathbb{H}^{n\times n}$ is unknown. The general solution of (5.17) is found to be

$$\mathbf{X}(t) = \mathbf{B}\exp\left(-\mathbf{A}t\right)\left(\int \exp\left(\mathbf{A}t\right)dt\right).$$

If $\mathbf{A}$ is invertible, then by (5.12) the general solution of (5.17) is $\mathbf{X}(t) = \mathbf{B}\{\mathbf{A}^{-1} + \exp\left(-\mathbf{A}t\right)\mathbf{G}\}$. If $\mathbf{A}$ is noninvertible, then an explicit form for a general solution of (5.17) is

$$\mathbf{X}(t) =$$
$$\mathbf{B}\left\{\mathbf{A}^D + (\mathbf{I}-\mathbf{A}\mathbf{A}^D)t\left(\mathbf{I} - \frac{\mathbf{A}}{2}t + \frac{\mathbf{A}^2}{3!}t^2 + \dots(-1)^{k-1}\frac{\mathbf{A}^{k-1}}{k!}t^{k-1}\right) + \mathbf{G}\right\}.$$

If we put $\mathbf{G} = \mathbf{0}$, then we obtain the following partial solution of (5.17),

$$\mathbf{X}(t) = \mathbf{B}\mathbf{A}^D + (\mathbf{B}-\mathbf{B}\mathbf{A}\mathbf{A}^D)t - \frac{1}{2}(\mathbf{B}\mathbf{A}-\mathbf{B}\mathbf{A}^2\mathbf{A}^D)t^2 + \dots\frac{(-1)^{k-1}}{k!}(\mathbf{B}\mathbf{A}^{k-1}-\mathbf{B}\mathbf{A}^k\mathbf{A}^D)t^k. \tag{5.18}$$

**Theorem 5.3.** *If* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *has index* $k$ *and* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^k = r \le n$, *then the partial solution (5.18),* $\mathbf{X}(t) = (x_{ij})$, *possess the following determinantal representation,*

*1. when* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *is Hermitian,*

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left(\mathbf{A}^{k+1}_{j.}\left(\check{\mathbf{b}}^{(k)}_{.i}\right)\right)^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r,n}} |(\mathbf{A}^{k+1})^{\alpha}_{\alpha}|} + \left(b_{ij} - \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left(\mathbf{A}^{k+1}_{j.}\left(\check{\mathbf{b}}^{(k+1)}_{i.}\right)\right)^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r,n}} |(\mathbf{A}^{k+1})^{\alpha}_{\alpha}|}\right)t$$
$$-\frac{1}{2}\left(\check{b}^{(1)}_{ij} - \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left(\mathbf{A}^{k+1}_{j.}\left(\check{\mathbf{b}}^{(k+2)}_{i.}\right)\right)^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r,n}} |(\mathbf{A}^{k+1})^{\alpha}_{\alpha}|}\right)t^2 + \dots$$
$$\frac{(-1)^k}{k!}\left(\check{b}^{(k-1)}_{ij} - \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left(\mathbf{A}^{k+1}_{j.}\left(\check{\mathbf{b}}^{(2k)}_{i.}\right)\right)^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r,n}} |(\mathbf{A}^{k+1})^{\alpha}_{\alpha}|}\right)t^k,$$

*where* $\mathbf{BA}^l =: \check{\mathbf{B}}^{(l)} = (\check{b}_{ij}^{(l)}) \in \mathbb{H}^{n\times n}$ *for all* $l = \overline{k, 2k}$*;*

2. *when* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *is arbitrary,*

$$x_{ij} = \frac{\sum\limits_{s=1}^{n}\left(\sum\limits_{\alpha\in I_{r,\,n}\{s\}} \mathrm{rdet}_s\left(\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*\right)_{.s}(\check{\mathbf{d}}^{(0)})\right)_{\alpha}^{\alpha}\right) a_{sj}^{(k)}}{\sum\limits_{\alpha\in I_{r,\,n}} \left|\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*\right)_{\alpha}^{\alpha}\right|}$$

$$+\left(b_{ij} - \frac{\sum\limits_{s=1}^{n}\left(\sum\limits_{\alpha\in I_{r,\,n}\{s\}} \mathrm{rdet}_s\left(\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*\right)_{.s}(\check{\mathbf{d}}^{(1)})\right)_{\alpha}^{\alpha}\right) a_{sj}^{(k)}}{\sum\limits_{\alpha\in I_{r,\,n}} \left|\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*\right)_{\alpha}^{\alpha}\right|}\right) t$$

$$-\frac{1}{2}\left(\check{b}_{ij}^{(1)} - \frac{\sum\limits_{s=1}^{n}\left(\sum\limits_{\alpha\in I_{r,\,n}\{s\}} \mathrm{rdet}_s\left(\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*\right)_{.s}(\check{\mathbf{d}}^{(2)})\right)_{\alpha}^{\alpha}\right) a_{sj}^{(k)}}{\sum\limits_{\alpha\in I_{r,\,n}} \left|\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*\right)_{\alpha}^{\alpha}\right|}\right) t^2 + \ldots$$

$$\frac{(-1)^k}{k!}\left(\check{b}_{ij}^{(k-1)} - \frac{\sum\limits_{s=1}^{n}\left(\sum\limits_{\alpha\in I_{r,\,n}\{s\}} \mathrm{rdet}_s\left(\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*\right)_{.s}(\check{\mathbf{d}}^{(k)})\right)_{\alpha}^{\alpha}\right) a_{sj}^{(k)}}{\sum\limits_{\alpha\in I_{r,\,n}} \left|\left(\mathbf{A}^{2k+1}\left(\mathbf{A}^{2k+1}\right)^*\right)_{\alpha}^{\alpha}\right|}\right) t^k,$$

*where* $\mathbf{BA}^{k+l}(\mathbf{A}^{2k+1})^* = \mathbf{BA}^l\check{\mathbf{A}} =: \check{\mathbf{D}}^{(l)} = (\check{d}_{ij}^{(l)}) \in \mathbb{H}^{n\times n}$ *for all* $l = \overline{1, k}$ *and for all* $i, j = \overline{1, n}$*.*

*Proof.* The proof is similar to the proof of Theorem 5.2 by using the determinantal representation of the Drazin inverse (3.6) and (3.14), respectively. □

## 5.3. An Example

Let us consider the matrix equation

$$\mathbf{X}' + \mathbf{AX} = \mathbf{B}, \tag{5.19}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & k & -i \\ -k & 2 & j \\ i & -j & 1 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} i & j & k \\ 1 & -k & j \\ 1 & 0 & i \end{pmatrix}.$$

Since $\mathbf{A}^2 = \begin{pmatrix} 3 & 4k & -3i \\ -4k & 6 & 4j \\ 3i & -4j & 3 \end{pmatrix}$, $\det \mathbf{A} = \det \mathbf{A}^2 = 0$, and $\det \begin{pmatrix} 1 & k \\ -k & 2 \end{pmatrix} = 1$, $\det \begin{pmatrix} 3 & 4k \\ -4k & 6 \end{pmatrix} = 2$, then, by Theorem 2.11, $Ind\,\mathbf{A} = 1$ and $r_1 = \operatorname{rank} \mathbf{A} = 2$. Since $\mathbf{A}$ is Hermitian and $Ind\,\mathbf{A} = 1$, then we shall find the solutions $(x_{ij}) \in \mathbb{H}^{3\times 3}$ by (5.15),

$$x_{ij} = \frac{\sum\limits_{\beta \in J_{2,3}\{i\}} \operatorname{cdet}_i \left( \mathbf{A}^2_{.\,i} \left( \widehat{\mathbf{b}}^{(1)}_{.j} \right) \right) {}^{\beta}_{\beta}}{\sum\limits_{\beta \in J_{2,3}} \left| \left( \mathbf{A}^2 \right) {}^{\beta}_{\beta} \right|} + \left( b_{ij} - \frac{\sum\limits_{\beta \in J_{2,3}\{i\}} \operatorname{cdet}_i \left( \mathbf{A}^2_{.\,i} \left( \widehat{\mathbf{b}}^{(2)}_{.j} \right) \right) {}^{\beta}_{\beta}}{\sum\limits_{\beta \in J_{2,3}} \left| \left( \mathbf{A}^2 \right) {}^{\beta}_{\beta} \right|} \right) t$$

for all $i, j = 1, 2, 3$. We have, $\sum\limits_{\beta \in J_{2,3}} \left| \left( \mathbf{A}^2 \right) {}^{\beta}_{\beta} \right| = 4$,

$$\widehat{\mathbf{B}}^{(1)} = \mathbf{A}\mathbf{B} = \begin{pmatrix} k & 1+j & 1-i+k \\ 2 & i-2k & 1+2j-k \\ -j & i+k & 1+i-j \end{pmatrix},$$

$$\widehat{\mathbf{B}}^{(2)} = \mathbf{A}^2\mathbf{B} = \begin{pmatrix} 4k & 4+3j & 3-4i+3k \\ 6 & 4i-6k & 4+6j-4k \\ -4j & 4i+3k & 4+3i-3j \end{pmatrix}.$$

Therefore,

$$x_{11} = \frac{1}{4}\left( \operatorname{cdet}_1 \begin{pmatrix} k & 4k \\ 2 & 6 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} k & -3i \\ -j & 3 \end{pmatrix} \right) + \left( i - \frac{1}{4} \left[ \operatorname{cdet}_1 \begin{pmatrix} 4k & 4k \\ 6 & 6 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} 4k & -3i \\ -4j & 3 \end{pmatrix} \right] \right) t = \frac{1}{4}(-2k) + \left( i - \frac{1}{4}[0] \right) t = -0.5k + (i)t;$$

$$x_{12} = \frac{1}{4}\left( \operatorname{cdet}_1 \begin{pmatrix} 1+j & 4k \\ i-2k & 6 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} 1+j & -3i \\ i+k & 3 \end{pmatrix} \right) + \left( j - \frac{1}{4} \left[ \operatorname{cdet}_1 \begin{pmatrix} 4+3j & 4k \\ 4i-6k & 6 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} 4+3j & -3i \\ 4i+3k & 3 \end{pmatrix} \right] \right) t = \frac{1}{4}(-2+2j) + \left( j - \frac{1}{4}[2j] \right) t = -0.5 + 0.5j + (0.5j)\,t;$$

$$x_{13} = \frac{1}{4}\left( \operatorname{cdet}_1 \begin{pmatrix} 1-i+k & 4k \\ 1+2j-k & 6 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} 1-i+k & -3i \\ 1+ij & 3 \end{pmatrix} \right) + \left( k - \frac{1}{4} \left[ \operatorname{cdet}_1 \begin{pmatrix} 3-4i+3j & 4k \\ 4+6j-4k & 6 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} 3-4i+4k & -3i \\ 4+3i-3j & 3 \end{pmatrix} \right] \right) t = \frac{1}{4}(2+2i+2k) + \left( k - \frac{1}{4}[2+11k] \right) t = 0.5 + 0.5i + 0.5k + (-0.5 - 4.5k)\,t;$$

$$x_{21} = \frac{1}{4}\left(\mathrm{cdet}_2\begin{pmatrix}3 & k\\ -4k & 2\end{pmatrix} + \mathrm{cdet}_1\begin{pmatrix}2 & 4j\\ -j & 3\end{pmatrix}\right) + \left(1 - \frac{1}{4}\left[\mathrm{cdet}_2\begin{pmatrix}3 & 4k\\ -4k & 6\end{pmatrix} + \mathrm{cdet}_1\begin{pmatrix}6 & 4j\\ -4j & 3\end{pmatrix}\right]\right)t = \frac{1}{4}(4) + \left(1 - \frac{1}{4}\left[-4k\right]\right)t = 1 + (1+k)t;$$

$$x_{22} = \frac{1}{4}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1+j\\ -4k & i-2k\end{pmatrix} + \mathrm{cdet}_1\begin{pmatrix}i-2k & 4j\\ i+k & 3\end{pmatrix}\right) + \left(-k - \frac{1}{4}\left[\mathrm{cdet}_2\begin{pmatrix}3 & 4+3j\\ -4k & 4i-6k\end{pmatrix} + \mathrm{cdet}_1\begin{pmatrix}4i-6k & 4j\\ 4i+3k & 3\end{pmatrix}\right]\right)t = \frac{1}{4}(-2i-4k) + \left(-k - \frac{1}{4}\left[-4k\right]\right)t = -0.5i - k;$$

$$x_{23} = \frac{1}{4}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1-i+k\\ -4k & 1+2j-k\end{pmatrix} + \mathrm{cdet}_1\begin{pmatrix}1+2j-k & 4j\\ 1+i-j & 3\end{pmatrix}\right) + \left(j - \frac{1}{4}\left[\mathrm{cdet}_2\begin{pmatrix}3 & 3-4i+3k\\ -4k & 4+6j-4k\end{pmatrix} + \mathrm{cdet}_1\begin{pmatrix}4+6j-4k & 4j\\ 4+3i-3j & 3\end{pmatrix}\right]\right)t = \frac{1}{4}(-2+4j+2k) + \left(j - \frac{1}{4}\left[4j\right]\right)t = -0.5 + j + 0.5k;$$

$$x_{31} = \frac{1}{4}\left(\mathrm{cdet}_2\begin{pmatrix}3 & k\\ 3i & -j\end{pmatrix} + \mathrm{cdet}_2\begin{pmatrix}6 & 2\\ -4j & -j\end{pmatrix}\right) + \left(1 - \frac{1}{4}\left[\mathrm{cdet}_2\begin{pmatrix}3 & 4k\\ 3i & -4j\end{pmatrix} + \mathrm{cdet}_2\begin{pmatrix}6 & 6\\ -4j & -4j\end{pmatrix}\right]\right)t = \frac{1}{4}(2j) + \left(i - \frac{1}{4}\left[0\right]\right)t = 0.5j + t;$$

$$x_{32} = \frac{1}{4}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1+j\\ 3i & i+k\end{pmatrix} + \mathrm{cdet}_2\begin{pmatrix}6 & i-2k\\ -4j & i+k\end{pmatrix}\right) + \left(0 - \frac{1}{4}\left[\mathrm{cdet}_2\begin{pmatrix}3 & 4+3j\\ 3i & 4i+3k\end{pmatrix} + \mathrm{cdet}_2\begin{pmatrix}6 & 4i-6k\\ -4j & 4i+3k\end{pmatrix}\right]\right)t = \frac{1}{4}(-2i+2k) + \left(-\frac{1}{4}\left[2k\right]\right)t = -0.5i + 0.5k + (-0.5k)t;$$

$$x_{33} = \frac{1}{4}\left(\mathrm{cdet}_2\begin{pmatrix}3 & 1-i+k\\ 3i & 1+i-j\end{pmatrix} + \mathrm{cdet}_2\begin{pmatrix}6 & 1+2j-k\\ -4j & 1+i-j\end{pmatrix}\right) + \left(i - \frac{1}{4}\left[\mathrm{cdet}_2\begin{pmatrix}3 & 3-4i+3jk\\ 3i & 4+3i-3j\end{pmatrix} + \mathrm{cdet}_2\begin{pmatrix}6 & 4+6j-4k\\ -4j & 4+3i-3j\end{pmatrix}\right]\right)t = \frac{1}{4}(-2+2i-2j) + \left(i - \frac{1}{4}\left[2i-2j\right]\right)t = -0.5 + 0.5i - 0.5j + (0.5i-0.5j)t.$$

## 6. Determinantal Representations of the W-Weighted Drazin Inverse for an Arbitrary Matrix

The properties of the complex W-weighted Drazin inverse can be found in [1,44, 45, 46, 47, 48]. These properties can be generalized to $\mathbb{H}$. In particular, if $\mathbf{A} \in \mathbb{H}^{m\times n}$ with respect to $\mathbf{W} \in \mathbb{H}^{n\times m}$ and $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$, then

$$\mathbf{A}_{d,\mathbf{W}} = \mathbf{A}\left((\mathbf{WA})^D\right)^2 = \left((\mathbf{AW})^D\right)^2\mathbf{A}, \tag{6.1}$$

$$\mathbf{A}_{d,\mathbf{W}}\mathbf{W} = (\mathbf{WA})^D, \mathbf{WA}_{d,\mathbf{W}} = (\mathbf{AW})^D. \tag{6.2}$$

Determinantal representations W-weighted Drazin inverse of complex matrices have been received by a full-rank factorization in [37] and by a limit representation in [49].

Through the theory of column-row determinants, a determinantal representation W-weighted Drazin inverse over the quaternion skew-field for the first time has been obtained in [14] by the following theorem.

**Theorem 6.1.** *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{W} \in \mathbb{H}^{n\times m}$ *with* $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$ *and and* $\operatorname{rank}(\mathbf{AW})^k = s$. *Suppose that* $\mathbf{B} \in \mathbb{H}^{n\times(n-s)}_{n-s}$ *and* $\mathbf{C}^* \in \mathbb{H}^{m\times(m-s)}_{m-s}$ *are of full-ranks and*

$$\mathcal{R}_r(\mathbf{B}) = \mathcal{N}_r\left((\mathbf{WA})^k\right),\ \ \mathcal{N}_r(\mathbf{C}) = \mathcal{R}_r\left((\mathbf{AW})^k\right),$$
$$\mathcal{R}_l(\mathbf{C}) = \mathcal{N}_l\left((\mathbf{AW})^k\right),\ \ \mathcal{N}_l(\mathbf{B}) = \mathcal{R}_l\left((\mathbf{WA})^k\right).$$

*Denote*

$$\mathbf{M} = \begin{bmatrix} \mathbf{WAW} & \mathbf{B} \\ \mathbf{C} & 0 \end{bmatrix}.$$

*Then the W-weighted Drazin inverse* $\mathbf{A}_{d,\mathbf{W}} = (a)_{ij} \in \mathbb{H}^{n\times m}$ *has the following determinantal representations:*

$$a_{ij} = \frac{\sum_{k=1}^{m+n-s} L_{ki} m^*_{kj}}{\det \mathbf{M}^*\mathbf{M}}, i = \overline{1,m}, j = \overline{1,n}, \tag{6.3}$$

*or*

$$a_{ij} = \frac{\sum_{k=1}^{m+n-s} m^*_{ik} R_{jk}}{\det \mathbf{MM}^*}, i = \overline{1,m}, j = \overline{1,n}, \tag{6.4}$$

*where* $L_{ij}$ *are the left* $(ij)$*-th cofactor of* $\mathbf{M}^*\mathbf{M}$ *and* $R_{ij}$ *are the right* $(ij)$*-th cofactor of* $\mathbf{MM}^*$, *respectively, for all* $i, j = \overline{1, m+n-s}$.

As can be seen, the auxiliary matrices $\mathbf{B}$ and $\mathbf{C}$ have been used in the determinantal representations (6.3) and (6.4). In this chapter we escape it. We shall derive determinantal representations of the W-weighted Drazin inverse of an arbitrary matrix $\mathbf{A} \in \mathbb{H}^{m\times n}$ with respect to $\mathbf{W} \in \mathbb{H}^{n\times m}$ by using the determinantal representations of the Drazin inverse, of the Moore-Penrose inverse, and the limit representation of the W-weighted Drazin inverse in some particular case.

## 6.1. Determinantal Representations of the W-Weighted Drazin Inverse by using Determinantal Representations of the Drazin Inverse

Let $\mathbf{A} \in \mathbb{H}^{m\times n}$ and $\mathbf{W} \in \mathbb{H}^{n\times m}$. Denote $\mathbf{WA} =: \mathbf{U} = (u_{ij}) \in \mathbb{H}^{n\times n}$ and $\mathbf{AW} =: \mathbf{V} = (v_{ij}) \in \mathbb{H}^{m\times m}$. Due to Theorem 3.6 for an arbitrary element of the Drazin inverse $\mathbf{U}^D$, we have the following determinantal representations,

$$u_{ij}^{D,1} = \frac{\sum\limits_{t=1}^{n} u_{it}^{(k)} \sum\limits_{\beta\in J_{r,\,n}\{t\}} \operatorname{cdet}_t\left(\left(\left(\mathbf{U}^{2k+1}\right)^*\left(\mathbf{U}^{2k+1}\right)_{.\,t}(\hat{\mathbf{u}}_{.\,j})\right)\,{}^{\beta}_{\beta}\right)}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\mathbf{U}^{2k+1}\right)^*\left(\mathbf{U}^{2k+1}\right)\,{}^{\beta}_{\beta}\right|} \tag{6.5}$$

or

$$u_{ij}^{D,2} = \frac{\sum\limits_{s=1}^{n}\left(\sum\limits_{\alpha\in I_{r,\,n}\{s\}} \operatorname{rdet}_s\left(\left(\mathbf{U}^{2k+1}\left(\mathbf{U}^{2k+1}\right)^*\right)_{.\,s}(\check{\mathbf{u}}_{i\,.})\right)\,{}^{\alpha}_{\alpha}\right) u_{sj}^{(k)}}{\sum\limits_{\alpha\in I_{r,\,n}} \left|\left(\mathbf{U}^{2k+1}\left(\mathbf{U}^{2k+1}\right)^*\right)\,{}^{\alpha}_{\alpha}\right|} \tag{6.6}$$

where $\hat{\mathbf{u}}_{.j}$ is the $j$-th column of $(\mathbf{U}^{2k+1})^*\mathbf{U}^k =: \hat{\mathbf{U}} = (\hat{u}_{ij}) \in \mathbb{H}^{n\times n}$, and $\check{\mathbf{u}}_{i.}$ is the $i$-th row of $\mathbf{U}^k(\mathbf{U}^{2k+1})^* =: \check{\mathbf{U}} = (\check{u}_{ij}) \in \mathbb{H}^{n\times n}$ for all $i, j = \overline{1, n}$, and $r = \operatorname{rank}\mathbf{U}^{k+1} = \operatorname{rank}\mathbf{U}^k$.

Then, by (6.1), we can obtain the following determinantal representations of $\mathbf{A}_{d,\mathbf{W}} = (a_{ij}^{d,\mathbf{W}}) \in \mathbb{H}^{m\times n}$,

$$a_{ij}^{d,\mathbf{W}} = \sum_{q=1}^{n} a_{iq}(u_{qj}^{D})^{(2)} \tag{6.7}$$

where

$$(u_{qj}^{D})^{(2)} = \sum_{p=1}^{n} u_{qp}^{D,l} u_{pj}^{D,f} \tag{6.8}$$

for all $l, f = \overline{1, 2}$. $u_{ij}^{D,1}$ and $u_{ij}^{D,2}$ are represented by (6.5) and (6.6), respectively.

Similarly, using $\mathbf{V} = (v_{ij}) \in \mathbb{H}^{m\times m}$, we have the following determinantal representations of $\mathbf{A}_{d,\mathbf{W}}$,

$$a_{ij}^{d,\mathbf{W}} = \sum_{q=1}^{m} (v_{iq}^{D})^{(2)} a_{qj}. \tag{6.9}$$

The first factor is one of the four possible equations

$$(v_{iq}^{D})^{(2)} = \sum_{p=1}^{m} v_{ip}^{D,l} v_{pq}^{D,f} \tag{6.10}$$

for all $l, f = \overline{1, 2}$. An element of the Drazin inverse $\mathbf{V}^D$ can be represented by

$$v_{ij}^{D,1} = \frac{\sum\limits_{t=1}^{m} v_{it}^{(k)} \sum\limits_{\beta\in J_{r,\,m}\{t\}} \operatorname{cdet}_t \left( \left( \mathbf{V}^{2k+1} \right)^* \left( \mathbf{V}^{2k+1} \right)_{.\,t} (\hat{\mathbf{v}}_{.\,j}) \right) {}^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{r,\,m}} \left| \left( \mathbf{V}^{2k+1} \right)^* \left( \mathbf{V}^{2k+1} \right) {}^{\beta}_{\beta} \right|} \tag{6.11}$$

or

$$v_{ij}^{D,2} = \frac{\sum\limits_{s=1}^{m} \left( \sum\limits_{\alpha\in I_{r,\,m}\{s\}} \operatorname{rdet}_s \left( \left( \mathbf{V}^{2k+1} \left( \mathbf{V}^{2k+1} \right)^* \right)_{.s} (\check{\mathbf{v}}_{i\,.}) \right) {}^{\alpha}_{\alpha} \right) v_{sj}^{(k)}}{\sum\limits_{\alpha\in I_{r,\,m}} \left| \left( \mathbf{V}^{2k+1} \left( \mathbf{V}^{2k+1} \right)^* \right) {}^{\alpha}_{\alpha} \right|}, \tag{6.12}$$

where $\hat{\mathbf{v}}_{.s}$ is the $s$-th column of $(\mathbf{V}^{2k+1})^* \mathbf{V}^k =: \hat{\mathbf{V}} = (\hat{v}_{ij}) \in \mathbb{H}^{m\times m}$, and $\check{\mathbf{v}}_{t.}$ is the $t$-th row of $\mathbf{V}^k (\mathbf{V}^{2k+1})^* =: \check{\mathbf{V}} = (\check{v}_{ij}) \in \mathbb{H}^{m\times m}$ for all $s, t = \overline{1, m}$, and $r = \operatorname{rank} \mathbf{V}^{k+1} = \operatorname{rank} \mathbf{V}^k$.

## 6.2. Determinantal Representations of the W-Weighted Drazin Inverse by using Determinantal Representations of the Moore-Penrose Inverse

Consider the general algebraic structures (GAS) of the matrices $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{W} \in \mathbb{H}^{n\times m}$, $\mathbf{A}^+ \in \mathbb{H}^{n\times m}$, $\mathbf{W}^+ \in \mathbb{H}^{m\times n}$ and $\mathbf{A}_{d,\mathbf{W}} \in \mathbb{H}^{m\times n}$ with $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$ (e.g., [44, 45, 46, 47]).

Let exist $\mathbf{L} \in \mathbb{H}^{m\times m}$ and $\mathbf{Q} \in \mathbb{H}^{n\times n}$ such that

$$\mathbf{A} = \mathbf{L}\begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}\mathbf{Q}^{-1}, \quad \mathbf{W} = \mathbf{Q}\begin{bmatrix} \mathbf{W}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{22} \end{bmatrix}\mathbf{L}^{-1}.$$

Then

$$\mathbf{A}^{+} = \mathbf{Q}\begin{bmatrix} \mathbf{A}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{L}^{-1}, \quad \mathbf{W}^{+} = \mathbf{L}\begin{bmatrix} \mathbf{W}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{Q}^{-1},$$

$$\mathbf{A}_{d,\mathbf{W}} = \mathbf{L}\begin{bmatrix} (\mathbf{W}_{11}\mathbf{A}_{11}\mathbf{W}_{11})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{Q}^{-1},$$

where $\mathbf{L}, \mathbf{Q}, \mathbf{A}_{11}, \mathbf{W}_{11}$ are invertible matrices, and $\mathbf{A}_{22}\mathbf{W}_{22}, \mathbf{W}_{22}\mathbf{A}_{22}$ are nilpotent matrices. Due to [47], the following theorem can be expanded to $\mathbb{H}$.

**Theorem 6.2.** *Let $\mathbf{A} \in \mathbb{H}^{m\times n}$ and $\mathbf{W} \in \mathbb{H}^{n\times m}$ such that $\mathbf{A}_{22}\mathbf{W}_{22}$ and $\mathbf{W}_{22}\mathbf{A}_{22}$ be nilpotent matrices of index $k$ in GAS form. Then the weighted Drazin inverse of $\mathbf{A}$ with respect to $\mathbf{W}$ can be written as matrix expression involving the Moore-Penrose inverse,*

$$\mathbf{A}_{d,\mathbf{W}} = \left\{(\mathbf{AW})^{k}\left[(\mathbf{AW})^{2k+1}\right]^{+}(\mathbf{AW})^{k}\right\}\mathbf{W}^{+}, \tag{6.13}$$

*where $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$.*

Similarly, the following theorem can be obtained.

**Theorem 6.3.** *Let $\mathbf{A} \in \mathbb{H}^{m\times n}$ and $\mathbf{W} \in \mathbb{H}^{n\times m}$ such that $\mathbf{A}_{22}\mathbf{W}_{22}$ and $\mathbf{W}_{22}\mathbf{A}_{22}$ be nilpotent matrices of index $k$ in GAS form. Then the W-weighted Drazin inverse of $\mathbf{A}$ with respect to $\mathbf{W}$ can be written as the following matrix expression,*

$$\mathbf{A}_{d,\mathbf{W}} = \mathbf{W}^{+}\left\{(\mathbf{WA})^{k}\left[(\mathbf{WA})^{2k+1}\right]^{+}(\mathbf{WA})^{k}\right\}, \tag{6.14}$$

*where $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$.*

*Proof.* Since $\mathbf{W}_{22}\mathbf{A}_{22}$ is a nilpotent matrix of index $k$, then due to GAS of $\mathbf{A}$, $\mathbf{W}$ and their generalized inverses, we have the following Jordan canonical forms,

$$\mathbf{WA} = \mathbf{Q}\begin{bmatrix} \mathbf{W}_{11}\mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{22}\mathbf{A}_{22} \end{bmatrix}\mathbf{Q}^{-1}, \quad (\mathbf{WA})^{k} = \mathbf{Q}\begin{bmatrix} (\mathbf{W}_{11}\mathbf{A}_{11})^{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{Q}^{-1},$$

$$\left[(\mathbf{WA})^{2k+1}\right]^{+}=\mathbf{Q}\begin{bmatrix}(\mathbf{W}_{11}\mathbf{A}_{11})^{-2k-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1}.$$

Simple computing of $\mathbf{W}^{+}\left\{(\mathbf{WA})^{k}\left[(\mathbf{WA})^{2k+1}\right]^{+}(\mathbf{WA})^{k}\right\}$ proves the theorem,

$$\begin{aligned}
&\mathbf{W}^{+}\left\{(\mathbf{WA})^{k}\left[(\mathbf{WA})^{2k+1}\right]^{+}(\mathbf{WA})^{k}\right\}=\\
&\mathbf{L}\begin{bmatrix}\mathbf{W}_{11}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}(\mathbf{W}_{11}\mathbf{A}_{11})^{k} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}(\mathbf{W}_{11}\mathbf{A}_{11})^{-2k-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}(\mathbf{W}_{11}\mathbf{A}_{11})^{k} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1}=\\
&\mathbf{L}\begin{bmatrix}\mathbf{W}_{11}^{-1}(\mathbf{W}_{11}\mathbf{A}_{11})^{k}(\mathbf{W}_{11}\mathbf{A}_{11})^{-2k-1}(\mathbf{W}_{11}\mathbf{A}_{11})^{k} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1}=\\
&\mathbf{L}\begin{bmatrix}\mathbf{W}_{11}^{-1}(\mathbf{W}_{11}\mathbf{A}_{11})^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1}=\\
&\mathbf{L}\begin{bmatrix}(\mathbf{W}_{11}\mathbf{A}_{11}\mathbf{W}_{11})^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1}=\mathbf{A}_{d,\mathbf{W}}.
\end{aligned}$$

□

Using (6.13), an entry $a_{ij}^{d,\mathbf{W}}$ of the W-weighted Drazin inverse $\mathbf{A}_{d,\mathbf{W}}$ can be obtained as follows

$$a_{ij}^{d,\mathbf{W}}=\sum_{s=1}^{m}\sum_{t=1}^{m}\sum_{l=1}^{m}v_{is}^{(k)}\left(v_{st}^{(2k+1)}\right)^{+}v_{tl}^{(k)}w_{lj}^{+} \tag{6.15}$$

for all $i=\overline{1,m}$, $j=\overline{1,n}$.

Denote by $\check{\mathbf{w}}_{t.}$ the $t$-th row of $\mathbf{V}^{k}\mathbf{W}^{*}=:\check{\mathbf{W}}=(\check{w}_{ij})\in\mathbb{H}^{m\times n}$ for all $t=\overline{1,m}$. It follows from $\sum\limits_{l}v_{tl}^{(k)}\mathbf{w}_{l.}^{*}=\check{\mathbf{w}}_{t.}$ and (3.12) that

$$\sum_{l=1}^{m}v_{tl}^{(k)}w_{lj}^{+}=\sum_{l=1}^{m}v_{tl}^{(k)}\cdot\frac{\sum\limits_{\alpha\in I_{r_1,\,n}\{j\}}\operatorname{rdet}_j\left(\mathbf{WW}^{*}\right)_{j.}\left(\mathbf{w}_{l.}^{*}\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r_1,\,n}}\left|(\mathbf{WW}^{*})_{\alpha}^{\alpha}\right|}=$$

$$\frac{\sum\limits_{\alpha\in I_{r_1,\,n}\{j\}}\operatorname{rdet}_j\left(\left(\mathbf{WW}^{*}\right)_{j.}\left(\check{\mathbf{w}}_{t.}\right)\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r_1,\,n}}\left|(\mathbf{WW}^{*})_{\alpha}^{\alpha}\right|}, \tag{6.16}$$

where $r_1 = \operatorname{rank}\mathbf{W}$. Similarly, denote by $\check{\mathbf{v}}_{i.}$ the $t$-th row of $\mathbf{V}^k(\mathbf{V}^{2k+1})^* =:$ $\check{\mathbf{V}} = (\check{v}_{ij}) \in \mathbb{H}^{m\times m}$ for all $t = \overline{1,m}$. It follows from $\sum\limits_s v_{is}^{(k)}\left(\mathbf{v}_{s.}^{(2k+1)}\right)^* = \check{\mathbf{v}}_{i.}$ and (3.12) that

$$\sum_{s=1}^{m} v_{is}^{(k)}\left(v_{st}^{(2k+1)}\right)^{+} = \sum_{s=1}^{m} v_{is}^{(k)} \cdot \frac{\sum\limits_{\alpha\in I_{r,\,m}\{t\}} \operatorname{rdet}_t\left(\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^*\right)_{t.}\left(\mathbf{v}_{s.}^{(2k+1)}\right)^*\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^*\right)_{\alpha}^{\alpha}\right|} =$$

$$\frac{\sum\limits_{\alpha\in I_{r,\,m}\{t\}} \operatorname{rdet}_t\left(\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^*\right)_{t.}\left(\check{\mathbf{v}}_{i\,.}\right)\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^*\right)_{\alpha}^{\alpha}\right|}, \tag{6.17}$$

where $r = \operatorname{rank}\mathbf{W}^{k+1} = \operatorname{rank}\mathbf{W}^{k}$. Using (6.16) and (6.17) in (6.15), we obtain the following determinantal representation of $\mathbf{A}_{d,\mathbf{W}}$ ,

$$a_{ij}^{d,\mathbf{W}} = \frac{\sum\limits_{t=1}^{m}\sum\limits_{\alpha\in I_{r,\,m}\{t\}} \operatorname{rdet}_t\left(\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^*\right)_{t.}\left(\check{\mathbf{v}}_{i\,.}\right)\right)_{\alpha}^{\alpha} \sum\limits_{\alpha\in I_{r_1,\,n}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{W}\mathbf{W}^*\right)_{j\,.}\left(\check{\mathbf{w}}_{t.}\right)\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^*\right)_{\alpha}^{\alpha}\right| \sum\limits_{\alpha\in I_{r_1,\,n}} \left|\left(\mathbf{W}\mathbf{W}^*\right)_{\alpha}^{\alpha}\right|} \tag{6.18}$$

Thus, we have proved the following theorem.

**Theorem 6.4.** *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $\mathbf{W} \in \mathbb{H}_{r_1}^{n\times m}$ *with* $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$ *and* $r = \operatorname{rank}(\mathbf{AW})^{k+1} = \operatorname{rank}(\mathbf{AW})^{k}$. *Then the W-weighted Drazin inverse of* $\mathbf{A}$ *with respect to* $\mathbf{W}$ *possesses the determinantal representation (6.18), where* $\mathbf{V} = \mathbf{AW}$, $\check{\mathbf{V}} = \mathbf{V}^k(\mathbf{V}^{2k+1})^*$, *and* $\check{\mathbf{W}} = \mathbf{V}^k\mathbf{W}^*$.

Similarly we have the following theorem.

**Theorem 6.5.** *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *and* $\mathbf{W} \in \mathbb{H}_{r_1}^{n\times m}$ *with* $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$ *and* $r = \operatorname{rank}(\mathbf{WA})^{k+1} = \operatorname{rank}(\mathbf{WA})^{k}$. *Then the W-weighted Drazin inverse of* $\mathbf{A}$ *with respect to* $\mathbf{W}$ *possesses the following determinantal representation,*

$$a_{ij}^{d,\mathbf{W}} = \frac{\sum\limits_{t=1}^{n}\sum\limits_{\beta\in J_{r_1,\,m}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{W}^*\mathbf{W}\right)_{.i}\left(\hat{\mathbf{w}}_{.t}\right)\right)_{\beta}^{\beta} \sum\limits_{\beta\in J_{r,\,n}\{t\}} \operatorname{cdet}_t\left(\left(\left(\mathbf{U}^{2k+1}\right)^*\mathbf{U}^{2k+1}\right)_{.t}\left(\hat{\mathbf{u}}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,m}} \left|\left(\mathbf{W}^*\mathbf{W}\right)_{\beta}^{\beta}\right| \sum\limits_{\beta\in J_{r,\,n}} \left|\left(\left(\mathbf{U}^{2k+1}\right)^*\mathbf{U}^{2k+1}\right)_{\beta}^{\beta}\right|} \tag{6.19}$$

*where* $\mathbf{U} = \mathbf{W}\mathbf{A}$, $\hat{\mathbf{U}} = (\mathbf{U}^{2k+1})^*\mathbf{U}^k$, *and* $\hat{\mathbf{W}} = \mathbf{W}^*\mathbf{U}^k$.

*Proof.* Using (6.14), an entry $a_{ij}^{d,\mathbf{W}}$ of the W-weighted Drazin inverse $\mathbf{A}_{d,\mathbf{W}}$ can be obtained as follows

$$a_{ij}^{d,\mathbf{W}} = \sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{l=1}^{n} w_{is}^{+}u_{st}^{(k)}\left(u_{tl}^{(2k+1)}\right)^{+}u_{lj}^{(k)} \tag{6.20}$$

for all $i = \overline{1,m}$, $j = \overline{1,n}$. Denote by $\hat{\mathbf{w}}_{.t}$ the $t$-th column of $\mathbf{W}^*\mathbf{U}^k =: \hat{\mathbf{W}} = (\hat{w}_{ij}) \in \mathbb{H}^{m\times n}$ for all $t = \overline{1,n}$. It follows from $\sum\limits_{t} \mathbf{w}_{.s}^{*}u_{st}^{(k)} = \hat{\mathbf{w}}_{.t}$ and (3.11) that

$$\sum_{s=1}^{n} w_{is}^{+}u_{st}^{(k)} = \sum_{s=1}^{n} \frac{\sum\limits_{\beta\in J_{r_1,\,m}\{i\}} \operatorname{cdet}_i\left(\mathbf{W}^*\mathbf{W}\right)_{.i}\left(\mathbf{w}_{.s}^{*}\right)\,{}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,m}} \left|\left(\mathbf{W}^*\mathbf{W}\right)\,{}_{\beta}^{\beta}\right|}\cdot u_{st}^{(k)} =$$

$$\frac{\sum\limits_{\beta\in J_{r_1,\,m}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{W}^*\mathbf{W}\right)_{.i}\left(\hat{\mathbf{w}}_{.t}\right)\right)\,{}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,m}} \left|\left(\mathbf{W}^*\mathbf{W}\right)\,{}_{\beta}^{\beta}\right|}, \tag{6.21}$$

where $r_1 = \operatorname{rank}\mathbf{W}$. Similarly, denote by $\hat{\mathbf{u}}_{.j}$ the $j$-th column of $(\mathbf{U}^{2k+1})^*\mathbf{U}^k =: \hat{\mathbf{U}} = (\hat{u}_{ij}) \in \mathbb{H}^{n\times n}$ for all $j = \overline{1,n}$. It follows from $\sum\limits_{l}\left(\mathbf{u}_{.l}^{(2k+1)}\right)^{*}u_{lj}^{(k)} = \hat{\mathbf{u}}_{.j}$ and (3.11) that

$$\sum_{l=1}^{n}\left(u_{tl}^{(2k+1)}\right)^{+}u_{lj}^{(k)} =$$

$$\sum_{l=1}^{n} \frac{\sum\limits_{\beta\in J_{r,\,n}\{t\}} \operatorname{cdet}_t\left(\left(\left(\mathbf{U}^{2k+1}\right)^{*}\mathbf{U}^{2k+1}\right)_{.t}\left(\mathbf{u}_{.l}^{(2k+1)}\right)^{*}\right)\,{}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\left(\mathbf{U}^{2k+1}\right)^{*}\mathbf{U}^{2k+1}\right)\,{}_{\beta}^{\beta}\right|}\cdot u_{lj}^{(k)} =$$

$$\frac{\sum\limits_{\beta\in J_{r,\,n}\{t\}} \operatorname{cdet}_t\left(\left(\left(\mathbf{U}^{2k+1}\right)^{*}\mathbf{U}^{2k+1}\right)_{.t}\left(\hat{\mathbf{u}}_{.j}\right)\right)\,{}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,n}} \left|\left(\left(\mathbf{U}^{2k+1}\right)^{*}\mathbf{U}^{2k+1}\right)\,{}_{\beta}^{\beta}\right|}, \tag{6.22}$$

where $r = \operatorname{rank}(\mathbf{A}\mathbf{W})^{k+1} = \operatorname{rank}(\mathbf{A}\mathbf{W})^{k}$. Using the equations (6.22) and (6.21) in (6.20), we obtain (6.19). □

## 6.3. Determinantal Representations of the W-Weighted Drazin Inverse in Some Special Case

In this subsection we consider the determinantal representation of the W-weighted Drazin inverse of $\mathbf{A} \in \mathbb{H}^{m\times n}$ with respect to $\mathbf{W} \in \mathbb{H}^{n\times m}$ in a special case, when $\mathbf{AW} = \mathbf{V} = (v_{ij}) \in \mathbb{H}^{m\times m}$ and $\mathbf{WA} = \mathbf{U} = (u_{ij}) \in \mathbb{H}^{n\times n}$ are Hermitian. Then, for the determinantal representations of their Drazin inverse we can use (3.6) and (3.7).

For Hermitian matrix, we apply the method, which consists of the theorem on the limit representation of the Drazin inverse, lemmas on rank of matrices and on characteristic polynomial. By analogy to the complex case [39] we have the following limit representations of the W-weighted Drazin inverse,

$$\mathbf{A}_{d,W} = \lim_{\lambda\to 0}\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)^{-1}(\mathbf{AW})^k\mathbf{A} \tag{6.23}$$

and

$$\mathbf{A}_{d,W} = \lim_{\lambda\to 0}\mathbf{A}(\mathbf{WA})^k\left(\lambda \mathbf{I}_n + (\mathbf{WA})^{k+2}\right)^{-1} \tag{6.24}$$

where $\lambda \in \mathbb{R}_+$, and $\mathbb{R}_+$ is a set of the real positive numbers.

Denote by $\mathbf{v}_{.j}^{(k)}$ and $\mathbf{v}_{i.}^{(k)}$ the $j$-th column and the $i$-th row of $\mathbf{V}^k$, respectively. Denote by $\bar{\mathbf{V}}^k := (\mathbf{AW})^k\mathbf{A} \in \mathbb{H}^{m\times n}$ and $\bar{\mathbf{W}} = \mathbf{WAW} \in \mathbb{H}^{n\times m}$.

**Lemma 6.6.** *If* $\mathbf{AW} = \mathbf{V} = (v_{ij}) \in \mathbb{H}^{m\times m}$ *with* $Ind\mathbf{V} = k$*, then*

$$\operatorname{rank}\left(\mathbf{V}^{k+2}\right)_{.\,i}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right) \leq \operatorname{rank}\left(\mathbf{V}^{k+2}\right). \tag{6.25}$$

*Proof.* We have $\mathbf{V}^{k+2} = \bar{\mathbf{V}}^k\bar{\mathbf{W}}$. Let $\mathbf{P}_{i\,s}(-\bar{w}_{j\,s}) \in \mathbb{H}^{m\times m}$, $(s \neq i)$, be a matrix with $-\bar{w}_{j\,s}$ in the $(i,s)$-entry, 1 in all diagonal entries, and 0 in others. The matrix $\mathbf{P}_{i\,s}(-\bar{w}_{j\,s})$, $(s \neq i)$, is a matrix of an elementary transformation. It follows that

$$\left(\mathbf{V}^{k+2}\right)_{.\,i}\left(\bar{\mathbf{v}}_{.\,j}^{(k)}\right)\cdot\prod_{s\neq i}\mathbf{P}_{i\,s}(-\bar{w}_{j\,s}) = \begin{pmatrix} \sum\limits_{s\neq j}\bar{v}_{1s}^{(k)}\bar{w}_{s1} & \dots & \bar{v}_{1j}^{(k)} & \dots & \sum\limits_{s\neq j}\bar{v}_{1s}^{(k)}\bar{w}_{sm} \\ \dots & \dots & \dots & \dots & \dots \\ \sum\limits_{s\neq j}\bar{v}_{ms}^{(k)}\bar{w}_{s1} & \dots & \underset{i-th}{\bar{v}_{mj}^{(k)}} & \dots & \sum\limits_{s\neq j}\bar{v}_{ms}^{(k)}\bar{w}_{sm} \end{pmatrix}.$$

We have the next factorization of the obtained matrix.

$$\begin{pmatrix} \sum\limits_{s\neq j} \bar{v}_{1s}^{(k)}\bar{w}_{s1} & \dots & \bar{v}_{1j}^{(k)} & \dots & \sum\limits_{s\neq j} \bar{v}_{1s}^{(k)}\bar{w}_{sm} \\ \dots & \dots & \dots & \dots & \dots \\ \sum\limits_{s\neq j} \bar{v}_{ms}^{(k)}\bar{w}_{s1} & \dots & \underset{i-th}{\bar{v}_{mj}^{(k)}} & \dots & \sum\limits_{s\neq j} \bar{v}_{ms}^{(k)}\bar{w}_{sm} \end{pmatrix} =$$

$$= \begin{pmatrix} \bar{v}_{11}^{(k)} & \bar{v}_{12}^{(k)} & \dots & \bar{v}_{1n}^{(k)} \\ \bar{v}_{21}^{(k)} & \bar{v}_{22}^{(k)} & \dots & \bar{v}_{2n}^{(k)} \\ \dots & \dots & \dots & \dots \\ \bar{v}_{m1}^{(k)} & \bar{v}_{m2}^{(k)} & \dots & \bar{v}_{mn}^{(k)} \end{pmatrix} \begin{pmatrix} \bar{w}_{11} & \dots & 0 & \dots & \bar{w}_{1m} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \bar{w}_{n1} & \dots & \underset{i-th}{0} & \dots & \bar{w}_{nm} \end{pmatrix} j-th.$$

Denote $\tilde{\mathbf{W}} := \begin{pmatrix} \bar{w}_{11} & \dots & 0 & \dots & \bar{w}_{1m} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \bar{w}_{n1} & \dots & \underset{i-th}{0} & \dots & \bar{w}_{nm} \end{pmatrix} j-th.$ The matrix $\tilde{\mathbf{W}}$ is obtained from $\bar{\mathbf{W}} = \mathbf{WAW}$ by replacing all entries of the $j$-th row and the $i$th column with zeroes except for 1 in the $(i, j)$-entry. Since elementary transformations of a matrix do not change a rank, then $\operatorname{rank} \mathbf{V}_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right) \leq \min\left\{\operatorname{rank} \bar{\mathbf{V}}^k, \operatorname{rank} \tilde{\mathbf{W}}\right\}$. It is obvious that

$$\operatorname{rank} \bar{\mathbf{V}}^k = \operatorname{rank} (\mathbf{AW})^k\mathbf{A} \geq \operatorname{rank} (\mathbf{AW})^{k+2},$$
$$\operatorname{rank} \tilde{\mathbf{W}} \geq \operatorname{rank}\ \mathbf{WAW} \geq \operatorname{rank} (\mathbf{AW})^{k+2}.$$

From this the inequality (3.1) follows immediately. □

The next lemma is proved similarly.

**Lemma 6.7.** *If* $\mathbf{WA} = \mathbf{U} = (u_{ij}) \in \mathbb{H}^{n\times n}$ *with* $Ind\mathbf{U} = k$*, then*

$$\operatorname{rank} \left(\mathbf{U}^{k+2}\right)_{i\,.}\left(\bar{\mathbf{u}}_{j\,.}^{(k)}\right) \leq \operatorname{rank}\left(\mathbf{U}^{k+2}\right),$$

*where* $\bar{\mathbf{U}}^k := \mathbf{A}(\mathbf{WA})^k \in \mathbb{H}^{m\times n}$.

Analogues of the characteristic polynomial are considered in the following two lemmas.

**Lemma 6.8.** *If* $\mathbf{AW} = \mathbf{V} = (v_{ij}) \in \mathbb{H}^{m\times m}$ *is Hermitian with* $Ind\mathbf{V} = k$ *and* $\lambda \in \mathbb{R}$*, then*

$$\mathrm{cdet}_i \left(\lambda \mathbf{I}_m + \mathbf{V}^{k+2}\right)_{.\,i} \left(\bar{\mathbf{v}}_{.j}^{(k)}\right) = c_1^{(ij)}\lambda^{n-1} + c_2^{(ij)}\lambda^{n-2} + \ldots + c_n^{(ij)}, \quad (6.26)$$

*where* $c_n^{(ij)} = \mathrm{cdet}_i \left(\mathbf{V}^{k+2}\right)_{.\,i} \left(\bar{\mathbf{v}}_{.j}^{(k)}\right)$ *and*

$$c_s^{(ij)} = \sum_{\beta\in J_{s,\,n}\{i\}} \mathrm{cdet}_i \left(\left(\mathbf{V}^{k+2}\right)_{.\,i} \left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$$

*for all* $s = \overline{1, n-1}$*,* $i, j = \overline{1, n}$*.*

*Proof.* Consider the Hermitian matrix $\left(t\mathbf{I} + \mathbf{V}^{k+2}\right)_{.\,i} (\mathbf{v}_{.\,i}^{(k+2)}) \in \mathbb{H}^{n\times n}$. Taking into account Theorem 2.13, we obtain

$$\det \left(\lambda \mathbf{I} + \mathbf{V}^{k+2}\right)_{.\,i} \left(\mathbf{v}_{.\,i}^{(k+2)}\right) = d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_n, \qquad (6.27)$$

where $d_s = \sum\limits_{\beta\in J_{s,\,n}\{i\}} |\left(\mathbf{V}^{k+2}\right)_{\beta}^{\beta}|$ is the sum of all principal minors of order $s$ that contain the $i$-th column for all $s = \overline{1, n-1}$ and $d_n = \det\left(\mathbf{V}^{k+2}\right)$. Consequently, we have $\mathbf{v}_{.\,i}^{(k+2)} = \begin{pmatrix} \sum_l \bar{v}_{1l}^{(k)}\bar{w}_{li} \\ \sum_l \bar{v}_{2l}^{(k)}\bar{w}_{li} \\ \vdots \\ \sum_l \bar{v}_{nl}^{(k)}\bar{w}_{li} \end{pmatrix} = \sum_l \bar{\mathbf{v}}_{.\,l}^{(k)}\bar{w}_{li}$, where $\bar{\mathbf{v}}_{.\,l}^{(k)}$ is the $l$-th column of $\bar{\mathbf{V}}^k = (\mathbf{AW})^k\mathbf{A}$ and $\mathbf{WAW} = \bar{\mathbf{W}} = (\bar{w}_{li})$ for all $l = \overline{1, n}$. By Theorem 2.5, we obtain on the one hand

$$\det \left(\lambda \mathbf{I} + \mathbf{V}^{k+2}\right)_{.\,i} \left(\mathbf{v}_{.\,i}^{(k+2)}\right) = \mathrm{cdet}_i \left(\lambda \mathbf{I} + \mathbf{V}^{k+2}\right)_{.\,i} \left(\mathbf{v}_{.\,i}^{(k+2)}\right) =$$
$$= \sum_l \mathrm{cdet}_i \left(\lambda \mathbf{I} + \mathbf{V}^{k+2}\right)_{.\,l} \left(\bar{\mathbf{v}}_{.\,l}^{(k)}\bar{w}_{li}\right) = \sum_l \mathrm{cdet}_i \left(\lambda \mathbf{I} + \mathbf{V}^{k+2}\right)_{.\,i} \left(\bar{\mathbf{v}}_{.\,l}^{(k)}\right) \cdot \bar{w}_{li} \qquad (6.28)$$

On the other hand, having changed the order of summation, for all $s = \overline{1, n-1}$ we have

$$d_s = \sum_{\beta \in J_{s,\, n}\{i\}} \det \left(\mathbf{V}^{k+2}\right)_{\beta}^{\beta} = \sum_{\beta \in J_{s,\, n}\{i\}} \mathrm{cdet}_i \left(\mathbf{V}^{k+2}\right)_{\beta}^{\beta} =$$
$$\sum_{\beta \in J_{s,\, n}\{i\}} \sum_{l} \mathrm{cdet}_i \left(\left(\mathbf{V}^{k+2}\right)_{.\,i} \left(\bar{\mathbf{v}}_{.\,l}^{(k)} \bar{w}_{l\,i}\right)\right)_{\beta}^{\beta} =$$
$$\sum_{l} \sum_{\beta \in J_{s,\, n}\{i\}} \mathrm{cdet}_i \left(\left(\mathbf{V}^{k+2}\right)_{.\,i} \left(\bar{\mathbf{v}}_{.\,l}^{(k)}\right)\right)_{\beta}^{\beta} \cdot \bar{w}_{l\,i}. \quad (6.29)$$

By substituting (6.28) and (6.29) in (6.27), and equating factors at $\bar{w}_{l\,i}$ when $l = j$, we obtain (6.26). □

By analogy can be proved the following lemma.

**Lemma 6.9.** *If* $\mathbf{WA} = \mathbf{U} = (u_{ij}) \in \mathbb{H}^{n\times n}$ *is Hermitian with* $Ind\mathbf{U} = k$ *and* $\lambda \in \mathbb{R}$*, then*

$$\mathrm{rdet}_j(\lambda \mathbf{I} + \mathbf{U}^{k+2})_{j\,.}\,(\bar{\mathbf{u}}_{i.}^{(k)}) = r_1^{(ij)} \lambda^{n-1} + r_2^{(ij)} \lambda^{n-2} + \ldots + r_n^{(ij)},$$

*where* $r_s^{(ij)} = \sum\limits_{\alpha \in I_{s,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{U}^{k+2})_{j\,.}\,(\bar{\mathbf{u}}_{i.}^{(k)})\right)_{\alpha}^{\alpha}$ *and* $r_n^{(ij)} = \mathrm{rdet}_j(\mathbf{U}^{k+2})_{j\,.}\,(\bar{\mathbf{u}}_{i.}^{(k)})$ *for all* $s = \overline{1, n-1}$ *and* $i, j = \overline{1, n}$*.*

**Theorem 6.10.** *If* $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{W} \in \mathbb{H}^{n\times m}$*, and* $\mathbf{AW} \in \mathbb{H}^{m\times m}$ *is Hermitian with* $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$ *and* $\mathrm{rank}(\mathbf{AW})^{k+1} = \mathrm{rank}(\mathbf{AW})^{k} = r$*, then the W-weighted Drazin inverse* $\mathbf{A}_{d,W} = \left(a_{ij}^{d,W}\right) \in \mathbb{H}^{m\times n}$ *with respect to* $\mathbf{W}$ *possess the following determinantal representations:*

$$a_{ij}^{d,W} = \frac{\sum\limits_{\beta \in J_{r,\, m}\{i\}} \mathrm{cdet}_i \left((\mathbf{AW})_{.\,i}^{k+2} \left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r,\ m}} \left|(\mathbf{AW})^{k+2}\,{}_{\beta}^{\beta}\right|}, \quad (6.30)$$

*where* $\bar{\mathbf{v}}_{.j}^{(k)}$ *is the* $j$*-th column of* $\bar{\mathbf{V}}^{k} = (\mathbf{AW})^{k}\mathbf{A}$ *for all* $j = \overline{1, m}$*.*

*Proof.* The matrix $\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)^{-1} \in \mathbb{H}^{m\times m}$ is a full-rank Hermitian matrix. Taking into account Theorem 2.9 it has an inverse, which we represent as a left inverse matrix

$$\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)^{-1} = \frac{1}{\det\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)} \begin{pmatrix} L_{11} & L_{21} & \ldots & L_{m1} \\ L_{12} & L_{22} & \ldots & L_{m2} \\ \ldots & \ldots & \ldots & \ldots \\ L_{1m} & L_{2m} & \ldots & L_{mm} \end{pmatrix},$$

where $L_{ij}$ is a left $ij$-th cofactor of a matrix $\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}$. Then we have

$$\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)^{-1} (\mathbf{AW})^k \mathbf{A} =$$
$$= \frac{1}{\det\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)} \begin{pmatrix} \sum\limits_{s=1}^{m} L_{s1}\bar{v}_{s1}^{(k)} & \sum\limits_{s=1}^{m} L_{s1}\bar{v}_{s2}^{(k)} & \ldots & \sum\limits_{s=1}^{m} L_{s1}\bar{v}_{sn}^{(k)} \\ \sum\limits_{s=1}^{m} L_{s2}\bar{v}_{s1}^{(k)} & \sum\limits_{s=1}^{m} L_{s2}\bar{v}_{s2}^{(k)} & \ldots & \sum\limits_{s=1}^{m} L_{s2}\bar{v}_{sn}^{(k)} \\ \ldots & \ldots & \ldots & \ldots \\ \sum\limits_{s=1}^{m} L_{sm}\bar{v}_{s1}^{(k)} & \sum\limits_{s=1}^{m} L_{sm}\bar{v}_{s2}^{(k)} & \ldots & \sum\limits_{s=1}^{m} L_{sm}\bar{v}_{sn}^{(k)} \end{pmatrix}.$$

By (6.23) and using the definition of a left cofactor, we obtain

$$\mathbf{A}_{d,W} = \lim_{\alpha \to 0} \begin{pmatrix} \frac{\operatorname{cdet}_1\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)_{.1}\left(\bar{\mathbf{v}}_{.1}^{(k)}\right)}{\det\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)} & \ldots & \frac{\operatorname{cdet}_1\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)_{.1}\left(\bar{\mathbf{v}}_{.n}^{(k)}\right)}{\det\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)} \\ \ldots & \ldots & \ldots \\ \frac{\operatorname{cdet}_n\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)_{.n}\left(\bar{\mathbf{v}}_{.1}^{(k)}\right)}{\det\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)} & \ldots & \frac{\operatorname{cdet}_n\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)_{.m}\left(\bar{\mathbf{v}}_{.n}^{(k)}\right)}{\det\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)} \end{pmatrix}. \tag{6.31}$$

By Theorem 2.13, we have

$$\det\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right) = \lambda^m + d_1\lambda^{m-1} + d_2\lambda^{m-2} + \ldots + d_m,$$

where $d_s = \sum\limits_{\beta \in J_{s,\,m}} \left|\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)_{\beta}^{\beta}\right|$ is a sum of principal minors of $(\mathbf{AW})^{k+2}$ of order $s$ for all $s = \overline{1, m-1}$ and $d_m = \det(\mathbf{AW})^{k+2}$.

Since $\operatorname{rank}(\mathbf{AW})^{k+2} = \operatorname{rank}(\mathbf{AW})^{k+1} = \operatorname{rank}(\mathbf{AW})^{k} = r$, then $d_m = d_{m-1} = \ldots = d_{r+1} = 0$. It follows that $\det\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right) = \lambda^m + d_1\lambda^{m-1} + d_2\lambda^{m-2} + \ldots + d_r\lambda^{m-r}$.

Using (6.26) we have

$$\operatorname{cdet}_i\left(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}\right)_{.\,i}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right) = c_1^{(ij)}\lambda^{m-1} + c_2^{(ij)}\lambda^{m-2} + \ldots + c_m^{(ij)}$$

for $i = \overline{1, m}$ and $j = \overline{1, n}$, where $c_s^{(ij)} = \sum\limits_{\beta \in J_{s,\,m}\{i\}} \operatorname{cdet}_i\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$ for all $s = \overline{1, m-1}$ and $c_m^{(ij)} = \operatorname{cdet}_i(\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)$.

We shall prove that $c_k^{(ij)} = 0$, when $k \geq r+1$ for $i = \overline{1,m}$ and $j = \overline{1,n}$.

Since by Lemma 3.2 $\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right) \leq r$, then the matrix $\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)$ has no more $r$ right-linearly independent columns.

Consider $\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$, when $\beta \in J_{s,m}\{i\}$. It is a principal submatrix of $\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)$ of order $s \geq r+1$. Deleting both its $i$-th row and column, we obtain a principal submatrix of order $s-1$ of $(\mathbf{AW})^{k+2}$. We denote it by $\mathbf{M}$. The following cases are possible.

- Let $s = r+1$ and $\det \mathbf{M} \neq 0$. In this case all columns of $\mathbf{M}$ are right-linearly independent. The addition of all of them on one coordinate to columns of $\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$ keeps their right-linear independence. Hence, they are basis in a matrix $\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$, and the $i$-th column is the right linear combination of its basis columns. From this by Theorem 2.8, we get $\operatorname{cdet}_i\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta} = 0$, when $\beta \in J_{s,n}\{i\}$ and $s = r+1$.
- If $s = r+1$ and $\det \mathbf{M} = 0$, than $p$, $(p < s)$, columns are basis in $\mathbf{M}$ and in $\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$. Therefore, by Theorem 2.8, $\operatorname{cdet}_i\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta} = 0$ as well.
- If $s > r+1$, then $\det \mathbf{M} = 0$ and $p$, $(p < r)$, columns are basis in the both matrices $\mathbf{M}$ and $\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta}$. Therefore, by Theorem 2.8, we also have $\operatorname{cdet}_i\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta} = 0$.

Thus, in all cases we have $\operatorname{cdet}_i\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta} = 0$, when $\beta \in J_{s,m}\{i\}$ and $r+1 \leq s < m$. From here if $r+1 \leq s < m$, then

$$c_s^{(ij)} = \sum_{\beta \in J_{s,\,m}\{i\}} \operatorname{cdet}_i\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right)_{\beta}^{\beta} = 0,$$

and $c_m^{(ij)} = \operatorname{cdet}_i\left((\mathbf{AW})_{.i}^{k+2}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right)\right) = 0$ for all $i, j = \overline{1,n}$.

Hence, $\operatorname{cdet}_i\left(\lambda\mathbf{I} + (\mathbf{AW})^{k+2}\right)_{.i}\left(\bar{\mathbf{v}}_{.j}^{(k)}\right) = c_1^{(ij)}\lambda^{m-1} + c_2^{(ij)}\lambda^{m-2} + \ldots + c_r^{(ij)}\lambda^{m-r}$ for $i = \overline{1,m}$ and $j = \overline{1,n}$. By substituting these values in the matrix from (6.31), we obtain

$$\mathbf{A}_{d,W}=\lim_{\lambda\to 0}\begin{pmatrix}\frac{c_1^{(11)}\lambda^{m-1}+\ldots+c_r^{(11)}\lambda^{m-r}}{\lambda^m+d_1\lambda^{m-1}+\ldots+d_r\lambda^{m-r}} & \ldots & \frac{c_1^{(1n)}\lambda^{m-1}+\ldots+c_r^{(1n)}\lambda^{m-r}}{\lambda^m+d_1\lambda^{m-1}+\ldots+d_r\lambda^{m-r}}\\ \ldots & \ldots & \ldots\\ \frac{c_1^{(m1)}\lambda^{m-1}+\ldots+c_r^{(m1)}\lambda^{m-r}}{\lambda^m+d_1\lambda^{m-1}+\ldots+d_r\lambda^{m-r}} & \ldots & \frac{c_1^{(mn)}\lambda^{m-1}+\ldots+c_r^{(mn)}\lambda^{m-r}}{\lambda^m+d_1\lambda^{m-1}+\ldots+d_r\lambda^{m-r}}\end{pmatrix}=\begin{pmatrix}\frac{c_r^{(11)}}{d_r} & \ldots & \frac{c_r^{(1n)}}{d_r}\\ \ldots & \ldots & \ldots\\ \frac{c_r^{(m1)}}{d_r} & \ldots & \frac{c_r^{(mn)}}{d_r}\end{pmatrix}.$$

Here $c_r^{(ij)}=\sum\limits_{\beta\in J_{r,\,m}\{i\}}\operatorname{cdet}_i\left(\left(\mathbf{AW}\right)^{k+2}_{.\,i}\left(\bar{\mathbf{v}}^{(k)}_{.j}\right)\right)\,{}^{\beta}_{\beta}$ and $d_r=\sum\limits_{\beta\in J_{r,\,m}}\left|\left(\mathbf{AW}\right)^{k+2}\,{}^{\beta}_{\beta}\right|$. Thus, we have the determinantal representation of $\mathbf{A}_{d,W}$ by (6.30). □

The following theorem can be proved similarly.

**Theorem 6.11.** *If* $\mathbf{A}\in\mathbb{H}^{m\times n}$, $\mathbf{W}\in\mathbb{H}^{n\times m}$, *and* $\mathbf{WA}\in\mathbb{H}^{n\times n}$ *is Hermitian with* $k=\max\{Ind(\mathbf{AW}),Ind(\mathbf{WA})\}$ *and* $\operatorname{rank}(\mathbf{WA})^{k+1}=\operatorname{rank}(\mathbf{WA})^{k}=r$*, then the W-weighted Drazin inverse* $\mathbf{A}_{d,W}=\left(a_{ij}^{d,W}\right)\in\mathbb{H}^{m\times n}$ *with respect to* $\mathbf{W}$ *possess the following determinantal representations:*

$$a_{ij}^{d,W}=\frac{\sum\limits_{\alpha\in I_{r,n}\{j\}}\operatorname{rdet}_j\left(\left(\mathbf{WA}\right)^{k+2}_{j\,.}(\bar{\mathbf{u}}^{(k)}_{i.})\right)\,{}^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r,\,n}}\left|\left(\mathbf{WA}\right)^{k+2}\,{}^{\alpha}_{\alpha}\right|}. \tag{6.32}$$

*where* $\bar{\mathbf{u}}^{(k)}_{i.}$ *is the* $i$*-th row of* $\bar{\mathbf{U}}^{k}=\mathbf{A}(\mathbf{WA})^{k}$ *for all* $i=\overline{1,n}$.

## 6.4. An Example

Let us consider the matrices

$$\mathbf{A}=\begin{pmatrix}0 & i & 0\\ k & 1 & i\\ 1 & 0 & 0\\ 1 & -k & -j\end{pmatrix},\quad \mathbf{W}=\begin{pmatrix}k & 0 & i & 0\\ -j & k & 0 & 1\\ 0 & 1 & 0 & -k\end{pmatrix}.$$

Then

$$\mathbf{V} := \mathbf{AW} = \begin{pmatrix} -k & -j & 0 & i \\ -1-j & i+k & j & 1+j \\ k & 0 & i & 0 \\ -i+k & 1-j & i & i-k \end{pmatrix}, \ \mathbf{U} := \mathbf{WA} = \begin{pmatrix} i & j & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $\operatorname{rank}\mathbf{W} = 3$, $\operatorname{rank}\mathbf{V} = 3$, $\operatorname{rank}\mathbf{V}^3 = \operatorname{rank}\mathbf{V}^2 = 2$, $\operatorname{rank}\mathbf{U}^2 = \operatorname{rank}\mathbf{U} = 2$. Therefore, $Ind\,\mathbf{V} = 2$, $Ind\,\mathbf{U} = 1$, and $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\} = 2$.

It's evident that obtaining the W-weighted Drazin inverse of $\mathbf{A}$ with respect to $\mathbf{W}$ by using the matrix $\mathbf{U}$ by (6.19) is more convenient. We have

$$\mathbf{U}^2 = \begin{pmatrix} -1 & i+k & 0 \\ 0 & -1 & \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbf{U}^5 = \begin{pmatrix} i & 2+3j & 0 \\ 0 & k & \\ 0 & 0 & 0 \end{pmatrix}, \ (\mathbf{U}^5)^* = \begin{pmatrix} -i & 0 & 0 \\ 2-3j & -k & \\ 0 & 0 & 0 \end{pmatrix},$$

$$\left(\mathbf{U}^5\right)^* \mathbf{U}^5 = \begin{pmatrix} 1 & -2i-3k & 0 \\ 2i+3k & 14 & \\ 0 & 0 & 0 \end{pmatrix}, \hat{\mathbf{U}} = (\mathbf{U}^5)^*\mathbf{U}^2 = \begin{pmatrix} i & 1+j & 0 \\ -2+3j & -i+6k & \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{W}^* = \begin{pmatrix} -k & j & 0 \\ 0 & -k & 1 \\ -i & 0 & 0 \\ 0 & 1 & k \end{pmatrix}, \mathbf{W}^*\mathbf{W} = \begin{pmatrix} 2 & i & -j & j \\ -i & 2 & 0 & -2k \\ j & 0 & 1 & 0 \\ -j & 2k & 0 & 2 \end{pmatrix},$$

$$\hat{\mathbf{W}} = \mathbf{W}^*\mathbf{U}^2 = \begin{pmatrix} -k & 1-2j & 0 \\ 0 & i+k & 0 \\ i & 1+j & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

By (6.19),

$$a_{11}^{d,\mathbf{W}} = \frac{\sum\limits_{t=1}^{3} \sum\limits_{\beta \in I_{3,4}\{1\}} \operatorname{cdet}_1\left(\left(\mathbf{W}^*\mathbf{W}\right)_{.1}\left(\hat{\mathbf{w}}_{.t}\right)\right)_{\beta}^{\beta} \sum\limits_{\beta \in J_{2,3}\{t\}} \operatorname{cdet}_t\left(\left(\left(\left(\mathbf{U}^5\right)^* \mathbf{U}^5\right)_{.t}\left(\hat{\mathbf{u}}_{.1}\right)\right)_{\beta}^{\beta}\right)}{\sum\limits_{\beta \in J_{3,4}} \left|\left(\mathbf{W}^*\mathbf{W}\right)_{\beta}^{\beta}\right| \sum\limits_{\beta \in J_{2,3}} \left|\left(\left(\mathbf{U}^5\right)^* \mathbf{U}^5\right)_{\beta}^{\beta}\right|},$$

where

$$\sum_{\beta \in I_{3,4}\{1\}} \operatorname{cdet}_1\left(\left(\mathbf{W}^*\mathbf{W}\right)_{.1}\left(\hat{\mathbf{w}}_{.1}\right)\right)_{\beta}^{\beta} =$$

$$\operatorname{cdet}_1\begin{pmatrix} k & i & -j \\ 0 & 2 & 0 \\ i & 0 & 1 \end{pmatrix} + \operatorname{cdet}_1\begin{pmatrix} k & i & j \\ 0 & 2 & -2k \\ 0 & 2k & 1 \end{pmatrix} + \operatorname{cdet}_1\begin{pmatrix} k & -j & j \\ i & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 0,$$

$$\sum_{\beta\in I_{3,\,4}\{1\}} \mathrm{cdet}_1\left(\left(\mathbf{W}^*\mathbf{W}\right)_{.1}\left(\hat{\mathbf{w}}_{.2}\right)\right)_{\beta}^{\beta} = -2j,\ \sum_{\beta\in I_{3,\,4}\{1\}} \mathrm{cdet}_1\left(\left(\mathbf{W}^*\mathbf{W}\right)_{.1}\left(\hat{\mathbf{w}}_{.3}\right)\right)_{\beta}^{\beta} = 0,$$

$$\sum_{\beta\in J_{3,\,4}} \left|\left(\mathbf{W}^*\mathbf{W}\right)_{\beta}^{\beta}\right| = 2,$$

and

$$\sum_{\beta\in J_{2,\,3}\{1\}} \mathrm{cdet}_1\left(\left(\left(\mathbf{U}^5\right)^*\mathbf{U}^5\right)_{.1}\left(\hat{\mathbf{u}}_{.1}\right)\right)_{\beta}^{\beta} =$$

$$\mathrm{cdet}_1\begin{pmatrix} i & -2i-3k \\ -2+3j & 14 \end{pmatrix} + \mathrm{cdet}_1\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} = i,$$

$$\sum_{\beta\in J_{2,\,3}\{2\}} \mathrm{cdet}_2\left(\left(\left(\mathbf{U}^5\right)^*\mathbf{U}^5\right)_{.2}\left(\hat{\mathbf{u}}_{.1}\right)\right)_{\beta}^{\beta} = 0,$$

$$\sum_{\beta\in J_{2,\,3}\{3\}} \mathrm{cdet}_3\left(\left(\left(\mathbf{U}^5\right)^*\mathbf{U}^5\right)_{.3}\left(\hat{\mathbf{u}}_{.1}\right)\right)_{\beta}^{\beta} = 0,\ \sum_{\beta\in J_{2,\,3}} \left|\left(\left(\mathbf{U}^5\right)^*\mathbf{U}^5\right)_{\beta}^{\beta}\right| = 1.$$

Therefore,

$$a_{11}^{d,\mathbf{W}} = \frac{(0\cdot i) + (-2j\cdot 0) + (0\cdot 0)}{2\cdot 1} = 0.$$

Continuing in the same way, we finally get,

$$\mathbf{A}_{d,W} = \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ -1 & 5i-2k & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.33}$$

By (3.11), we obtain

$$\left(\mathbf{U}^5\right)^+ = \begin{pmatrix} -i & -3+2j & 0 \\ 0 & -k & \\ 0 & 0 & 0 \end{pmatrix},\ (\mathbf{AW})^D = \mathbf{U}^D = \mathbf{U}^2\left(\mathbf{U}^5\right)^+\mathbf{U}^2 = \begin{pmatrix} -i & -5 & 0 \\ 0 & -k & \\ 0 & 0 & 0 \end{pmatrix}.$$

We can verify (6.33) by (6.2). Indeed,

$$\mathbf{A}_{d,\mathbf{W}}\mathbf{W} = \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ -1 & 5i-2k & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} k & 0 & i & 0 \\ -j & k & 0 & 1 \\ 0 & 1 & 0 & -k \end{pmatrix} = \begin{pmatrix} -i & -5 & 0 \\ 0 & -k & \\ 0 & 0 & 0 \end{pmatrix} = (\mathbf{AW})^D.$$

We also obtain the W-weighted Drazin inverse of $\mathbf{A}$ with respect to $\mathbf{W}$ by (6.7), then we have

$$\mathbf{A}_{d,\mathbf{W}} = \mathbf{A}\left((\mathbf{WA})^D\right)^2 = \begin{pmatrix} 0 & -i & 0 \\ -k & 6+5i & 0 \\ -1 & 5i+5k & 0 \\ -1 & 5i+6k & 0 \end{pmatrix}, \tag{6.34}$$

The W-weighted Drazin inverse in (6.34) different from (6.33). It can be explained that the Jordan normal form of $\mathbf{WA}$ is unique only up to the order of the Jordan blocks. We get their complete equality, if $\mathbf{A}_{d,\mathbf{W}}$ from (6.34) be left-multiply by the nonsingular matrix $\mathbf{P}$ which is the product of multiplication of the following elementary matrices,

$$\mathbf{P} = \mathbf{P}_{2,4}(-k)\cdot\mathbf{P}_{4,3}(-1)\cdot\mathbf{P}_{3,4}(-6)\cdot\mathbf{P}_{4,1}(-j) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -k \\ 0 & 0 & 7 & -6 \\ -j & 0 & -1 & 1 \end{pmatrix}.$$

# 7. Cramer's Rule for the W-weighted Drazin Inverse Solution

## 7.1. Background of the Problem

In [46], Wei has established Cramer's rule for solving of a general restricted equation

$$\mathbf{WAWx} = \mathbf{b}, \;\; \mathbf{x} \in \mathcal{R}\left[(\mathbf{AW})^{k_1}\right], \tag{7.1}$$

where $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{W} \in \mathbb{C}^{n\times m}$ with $Ind\,(\mathbf{AW}) = k_1$, $Ind\,(\mathbf{WA}) = k_2$ and $\operatorname{rank}(\mathbf{AW})^{k_1} = r_1$, $\operatorname{rank}(\mathbf{WA})^{k_2} = r_2$. He proofed if $\mathbf{b} \in \mathcal{R}\left[(\mathbf{W})^{k_2}\mathbf{A}\right]$ and $r_1 = r_2$, then (7.1) has a unique solution, $\mathbf{x} = \mathbf{A}_{d,W}\mathbf{b}$, which can be presented by the following Cramer rule,

$$x_j = \det\begin{pmatrix} \mathbf{WAW}(j \longrightarrow \mathbf{b}) & \mathbf{U}_1 \\ \mathbf{V}_1(j \longrightarrow 0) & 0 \end{pmatrix} \Big/ \det\begin{pmatrix} \mathbf{WAW} & \mathbf{U}_1 \\ \mathbf{V}_1 & 0 \end{pmatrix}, \tag{7.2}$$

where $\mathbf{U}_1 \in \mathbb{C}^{n\times n-r_2}_{n-r_2}$, $\mathbf{V}_1^* \in \mathbb{C}^{m\times m-r_1}_{m-r_1}$ are matrices whose columns form bases for $\mathcal{N}((\mathbf{WA})^{k_2})$ and $\mathcal{N}((\mathbf{AW})^{k_1})$, respectively.

Recently, within the framework of the theory of column-row determinants Song [14] has considered a characterization of the W-weighted Drazin inverse over the quaternion skew and presented Cramer's rule of the restricted matrix equation,

$$\mathbf{W}_1\mathbf{A}\mathbf{W}_1\mathbf{X}\mathbf{W}_2\mathbf{B}\mathbf{W}_2 = \mathbf{D}, \tag{7.3}$$

$$\begin{array}{ll} \mathcal{R}_r(\mathbf{X}) \subset \mathcal{R}_r\left((\mathbf{A}\mathbf{W}_1)^{k_1}\right) & \mathcal{N}_r(\mathbf{X}) \supset \mathcal{N}_r\left((\mathbf{W}_2\mathbf{B})^{k_2}\right), \\ \mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l\left((\mathbf{B}\mathbf{W}_2)^{k_2}\right), & \mathcal{N}_l(\mathbf{X}) \supset \mathcal{N}_l\left((\mathbf{W}_1\mathbf{A})^{k_1}\right), \end{array} \tag{7.4}$$

where $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{W}_1 \in \mathbb{H}^{n\times m}$, $\mathbf{B} \in \mathbb{H}^{p\times q}$, $\mathbf{W}_2 \in \mathbb{H}^{q\times p}$, and $\mathbf{D} \in \mathbb{H}^{n\times p}$ with $k_1 = \max\{Ind(\mathbf{A}\mathbf{W}_1), Ind\,(\mathbf{W}_1\mathbf{A})\}$, $k_2 = \max\{Ind(\mathbf{B}\mathbf{W}_2), Ind\,(\mathbf{W}_2\mathbf{B})\}$, and $\operatorname{rank}(\mathbf{A}\mathbf{W}_1)^{k_1} = s_1$, $\operatorname{rank}(\mathbf{B}\mathbf{W}_2)^{k_2} = s_2$. He proved that if

$$\mathcal{R}_r(\mathbf{D}) \in \mathcal{R}_r\left((\mathbf{W}_1\mathbf{A})^{k_1}, (\mathbf{W}_2\mathbf{B})^{k_2}\right),\ \mathcal{R}_l(\mathbf{D}) \in \mathcal{R}_l\left((\mathbf{A}\mathbf{W}_1)^{k_1}, (\mathbf{B}\mathbf{W}_2)^{k_2}\right)$$

and there exist auxiliary matrices of full column rank, $\mathbf{L}_1 \in \mathbb{H}^{n\times n-s_1}_{n-s_1}$, $\mathbf{M}_1^* \in \mathbb{H}^{m\times m-s_1}_{m-s_1}$, $\mathbf{L}_2 \in \mathbb{H}^{q\times q-s_2}_{q-s_2}$, $\mathbf{M}_2^* \in \mathbb{H}^{p\times p-s_2}_{p-s_2}$ with additional terms of their ranges and null spaces, then the restricted matrix equation (7.3) has a unique solution,

$$\mathbf{X} = \mathbf{A}_{d,\mathbf{W}_1}\mathbf{D}\mathbf{B}_{d,\mathbf{W}_2}.$$

Using auxiliary matrices, $\mathbf{L}_1$, $\mathbf{M}_1$, $\mathbf{L}_2$, $\mathbf{M}_2$, Song presented its Cramer's rule by analogy to (7.2). In this chapter we avoid such approach and obtain explicit formulas for determinantal representations of the W-weighted Drazin inverse solutions of matrix equations by using only given matrices.

## 7.2. Cramer's Rules for the W-weighted Drazin Inverse Solutions of Some Matrix Equations

Consider the matrix equation (7.3) with the constraints (7.4). Denote $\mathbf{A}\mathbf{D}\mathbf{B} =: \tilde{\mathbf{D}} = \left(\tilde{d}_{lf}\right) \in \mathbb{H}^{m\times q}$, and $\bar{\mathbf{V}}\mathbf{D}\bar{\mathbf{U}} =: \bar{\mathbf{D}} = \left(\bar{d}_{lf}\right) \in \mathbb{H}^{m\times q}$, where $\bar{\mathbf{V}} := (\mathbf{A}\mathbf{W}_1)^{k_1}\mathbf{A}$, $\bar{\mathbf{U}} := \mathbf{B}(\mathbf{W}_2\mathbf{B})^{k_2}$.

**Theorem 7.1.** *Suppose* $\mathbf{D} \in \mathbb{H}^{n\times p}$, $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{W}_1 \in \mathbb{H}^{n\times m}_{r_1}$ *with* $k_1 = \max\{Ind(\mathbf{A}\mathbf{W}_1), Ind\,(\mathbf{W}_1\mathbf{A})\}$, *and* $\mathbf{B} \in \mathbb{H}^{p\times q}$, $\mathbf{W}_2 \in \mathbb{H}^{q\times p}_{r_2}$ *with* $k_2 = \max\{Ind(\mathbf{B}\mathbf{W}_2), Ind\,(\mathbf{W}_2\mathbf{B})\}$, *where* $\operatorname{rank}(\mathbf{A}\mathbf{W}_1)^{k_1} = s_1$, $\operatorname{rank}(\mathbf{B}\mathbf{W}_2)^{k_2} = s_2$. *If* $\mathcal{R}_r(\mathbf{D}) \in \mathcal{R}_r\left((\mathbf{W}_1\mathbf{A})^{k_1}, (\mathbf{W}_2\mathbf{B})^{k_2}\right)$, $\mathcal{R}_l(\mathbf{D}) \in \mathcal{R}_l\left((\mathbf{A}\mathbf{W}_1)^{k_1}, (\mathbf{B}\mathbf{W}_2)^{k_2}\right)$, *then the restricted matrix equation (7.3) has a unique solution,*

$$\mathbf{X} = \mathbf{A}_{d,\mathbf{W}_1}\mathbf{D}\mathbf{B}_{d,\mathbf{W}_2}, \tag{7.5}$$

*which possess the following determinantal representations for all* $i = \overline{1, m}$, $j = \overline{1, q}$.

*i)*

$$x_{ij} = \sum_{l=1}^{m} \sum_{f=1}^{q} (v_{il}^{D})^{(2)} \tilde{d}_{lf} (u_{fj}^{D})^{(2)}, \tag{7.6}$$

*where* $(v_{il}^{D}) = \mathbf{V}^{D}$ *is the Drazin inverse of* $\mathbf{V} = \mathbf{AW}_1$ *and* $(v_{il}^{D})^{(2)}$ *can be obtained by (6.10), and* $(u_{fj}^{D}) = \mathbf{U}^{D}$ *is the Drazin inverse of* $\mathbf{U} = \mathbf{W}_2\mathbf{B}$ *and* $(u_{qj}^{D})^{(2)}$ *can be obtained by (6.8).*

*ii) If* $\mathbf{AW}_1 \in \mathbb{H}^{m\times m}$ *and* $\mathbf{W}_2\mathbf{B} \in \mathbb{H}^{q\times q}$ *are Hermitian, then*

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{s_1,\, m}\{i\}} \operatorname{cdet}_i \left( (\mathbf{AW}_1)^{k_1+2}_{.\,i} \left(\mathbf{d}^{\mathbf{B}}_{.\,j}\right) \right) {}^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{s_1,\, m}} \left| (\mathbf{AW}_1)^{k_1+2} {}^{\beta}_{\beta} \right| \sum\limits_{\alpha\in I_{s_2,\, q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2} {}^{\alpha}_{\alpha} \right|}, \tag{7.7}$$

*or*

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{s_2,q}\{j\}} \operatorname{rdet}_j \left( (\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.} (\mathbf{d}^{\mathbf{A}}_{i\,.}) \right) {}^{\alpha}_{\alpha}}{\sum\limits_{\beta\in J_{s_1,\, m}} \left| (\mathbf{AW}_1)^{k_1+2} {}^{\beta}_{\beta} \right| \sum\limits_{\alpha\in I_{s_2,\, q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2} {}^{\alpha}_{\alpha} \right|}, \tag{7.8}$$

*where*

$$\mathbf{d}^{\mathbf{B}}_{.\,j} = \left( \sum_{\alpha\in I_{s_2,q}\{j\}} \operatorname{rdet}_j \left( (\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.} (\bar{\mathbf{d}}_{t.}) \right)^{\alpha}_{\alpha} \right) \in \mathbb{H}^{n\times 1}, \quad t = \overline{1, n} \tag{7.9}$$

$$\mathbf{d}^{\mathbf{A}}_{i\,.} = \left( \sum_{\beta\in J_{s_1,\, m}\{i\}} \operatorname{cdet}_i \left( (\mathbf{AW}_1)^{k_1+2}_{.\,i} \left(\bar{\mathbf{d}}_{.\,l}\right) \right) {}^{\beta}_{\beta} \right) \in \mathbb{H}^{1\times q}, \quad l = \overline{1, q} \tag{7.10}$$

*are the column vector and the row vector, respectively.* $\bar{\mathbf{d}}_{i.}$ *and* $\bar{\mathbf{d}}_{.j}$ *are the i-th row and the j-th column of* $\bar{\mathbf{D}}$ *for all* $i = \overline{1, n}$, $j = \overline{1, p}$.

*Proof.* The existence and uniqueness of the solution (7.5) can be proved similar as in ( [14], Theorem 5.2).

To derive Cramer's rule (7.6) we use (6.1). Then, we obtain

$$\mathbf{X} = \left((\mathbf{AW}_1)^{D})\right)^2 \mathbf{ADB} \left((\mathbf{W}_2\mathbf{B})^{D})\right)^2. \tag{7.11}$$

Denote $\mathbf{ADB} =: \tilde{\mathbf{D}} = \left(\tilde{d}_{lf}\right) \in \mathbb{H}^{m\times q}$, $\mathbf{V} := \mathbf{AW}_1$, and $\mathbf{U} := \mathbf{W}_2\mathbf{B}$. The equation (7.11) can be written component-wise as follows

$$x_{ij} = \sum_{s=1}^{p}\sum_{t=1}^{n}(a_{it}^{d,W1})d_{ts}(b_{sj}^{d,W_2}) = \sum_{s=1}^{p}\sum_{t=1}^{n}\left(\sum_{l=1}^{m}(v_{il}^{D})^{(2)}a_{lt}\right)d_{ts}\left(\sum_{f=1}^{q}b_{sf}(u_{fj}^{D})^{(2)}\right)$$

By changing the order of summation, from here it follows (7.6).

ii) If $\mathbf{A} \in \mathbb{H}_{r_1}^{m\times n}$, $\mathbf{B} \in \mathbb{H}_{r_2}^{p\times q}$ and $\mathbf{AW}_1 \in \mathbb{H}^{m\times m}$ and $\mathbf{W}_2\mathbf{B} \in \mathbb{H}^{q\times q}$ are Hermitian, then by Theorems 6.10 and 6.11 the W-weighted Drazin inverses $\mathbf{A}_{d,W_1} = \left(a_{ij}^{d,W_1}\right) \in \mathbb{H}^{m\times n}$ and $\mathbf{B}_{d,W_2} = \left(b_{ij}^{d,W_2}\right) \in \mathbb{H}^{q\times p}$ posses the following determinantal representations respectively,

$$a_{ij}^{d,W_1} = \frac{\sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{AW_1}\right)^{k_1+2}_{.\,i}\left(\bar{\mathbf{v}}_{.j}\right)\right)\,{}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,m}}\left|\left(\mathbf{AW_1}\right)^{k_1+2}\,{}_{\beta}^{\beta}\right|}, \tag{7.12}$$

where $\bar{\mathbf{v}}_{.j}$ is the $j$-th column of $\bar{\mathbf{V}} = (\mathbf{AW_1})^{k_1}\mathbf{A}$ for all $j = \overline{1,m}$, and

$$b_{ij}^{d,W_2} = \frac{\sum\limits_{\alpha\in I_{s_2,q}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{W_2B}\right)^{k_2+2}_{j\,.}(\bar{\mathbf{u}}_{i.})\right)\,{}_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{s_2,\,q}}\left|\left(\mathbf{W_2B}\right)^{k_2+2}\,{}_{\alpha}^{\alpha}\right|}, \tag{7.13}$$

where $\bar{\mathbf{u}}_{i.}$ is the $i$-th row of $\bar{\mathbf{U}} = \mathbf{B}(\mathbf{W_2B})^{k_2}$ for all $i = \overline{1,p}$. By component-wise writing (7.5) we obtain,

$$x_{ij} = \sum_{s=1}^{p}\left(\sum_{t=1}^{n}a_{it}^{d,W1}d_{ts}\right)\cdot b_{sj}^{d,W_2} \tag{7.14}$$

Denote by $\hat{\mathbf{d}}_{.s}$ the $s$-th column of $\bar{\mathbf{V}}\mathbf{D} = (\mathbf{AW_1})^{k_1}\mathbf{AD} =: \hat{\mathbf{D}} = (\hat{d}_{ij}) \in \mathbb{H}^{m\times p}$

for all $s = \overline{1,p}$. It follows from $\sum\limits_t \bar{\mathbf{v}}_{.\,t} d_{ts} = \hat{\mathbf{d}}_{.\,s}$ that

$$\sum_{t=1}^{n} a_{it}^{d,W1} d_{ts} = \sum_{t=1}^{n} \frac{\sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \operatorname{cdet}_i \left( (\mathbf{AW}_1)^{k_1+2}_{.\,i} \left( \bar{\mathbf{v}}_{.t} \right) \right) {}^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{s_1,\,m}} \left| (\mathbf{AW}_1)^{k_1+2} {}^{\beta}_{\beta} \right|} \cdot d_{ts} =$$

$$\frac{\sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \sum\limits_{t=1}^{n} \operatorname{cdet}_i \left( (\mathbf{AW}_1)^{k_1+2}_{.\,i} \left( \bar{\mathbf{v}}_{.t} \right) \right) {}^{\beta}_{\beta} \cdot d_{ts}}{\sum\limits_{\beta\in J_{s_1,\,m}} \left| (\mathbf{AW}_1)^{k_1+2} {}^{\beta}_{\beta} \right|} = \frac{\sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \operatorname{cdet}_i \left( (\mathbf{AW}_1)^{k_1+2}_{.\,i} \left( \hat{\mathbf{d}}_{.\,s} \right) \right) {}^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{s_1,\,m}} \left| (\mathbf{AW}_1)^{k_1+2} {}^{\beta}_{\beta} \right|} \tag{7.15}$$

Suppose $\mathbf{e}_{s.}$ and $\mathbf{e}_{.\,s}$ are respectively the unit row-vector and the unit column-vector whose components are $0$, except the $s$-th components, which are $1$. Substituting (7.15) and (7.13) in (7.14), we obtain

$$x_{ij} = \sum_{s=1}^{p} \frac{\sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \operatorname{cdet}_i \left( (\mathbf{AW}_1)^{k_1+2}_{.\,i} \left( \hat{\mathbf{d}}_{.\,s} \right) \right) {}^{\beta}_{\beta}}{\sum\limits_{\beta\in J_{s_1,\,m}} \left| (\mathbf{AW}_1)^{k_1+2} {}^{\beta}_{\beta} \right|} \frac{\sum\limits_{\alpha\in I_{s_2,q}\{j\}} \operatorname{rdet}_j \left( (\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.} (\bar{\mathbf{u}}_{s.}) \right) {}^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{s_2,\,q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2} {}^{\alpha}_{\alpha} \right|}.$$

Since

$$\hat{\mathbf{d}}_{.\,s} = \sum_{t=1}^{n} \mathbf{e}_{.\,t} \hat{d}_{ts},\ \bar{\mathbf{u}}_{s.} = \sum_{l=1}^{q} \bar{u}_{sl} \mathbf{e}_{l.},\ \sum_{s=1}^{p} \hat{d}_{ts} \bar{u}_{sl} = \bar{d}_{tl}, \tag{7.16}$$

then we have

$$x_{ij} =$$

$$\frac{\sum\limits_{s=1}^{p} \sum\limits_{t=1}^{n} \sum\limits_{l=1}^{q} \sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \operatorname{cdet}_i \left( (\mathbf{AW}_1)^{k_1+2}_{.\,i} \left( \mathbf{e}_{.\,t} \right) \right) {}^{\beta}_{\beta} \hat{d}_{ts} \bar{u}_{sl} \sum\limits_{\alpha\in I_{s_2,q}\{j\}} \operatorname{rdet}_j \left( (\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.} (\mathbf{e}_{l.}) \right) {}^{\alpha}_{\alpha}}{\sum\limits_{\beta\in J_{s_1,\,m}} \left| (\mathbf{AW}_1)^{k_1+2} {}^{\beta}_{\beta} \right| \sum\limits_{\alpha\in I_{s_2,\,q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2} {}^{\alpha}_{\alpha} \right|} =$$

$$\frac{\sum\limits_{t=1}^{n} \sum\limits_{l=1}^{q} \sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \operatorname{cdet}_i \left( (\mathbf{AW}_1)^{k_1+2}_{.\,i} \left( \mathbf{e}_{.\,t} \right) \right) {}^{\beta}_{\beta} \bar{d}_{tl} \sum\limits_{\alpha\in I_{s_2,q}\{j\}} \operatorname{rdet}_j \left( (\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.} (\mathbf{e}_{l.}) \right) {}^{\alpha}_{\alpha}}{\sum\limits_{\beta\in J_{s_1,\,m}} \left| (\mathbf{AW}_1)^{k_1+2} {}^{\beta}_{\beta} \right| \sum\limits_{\alpha\in I_{s_2,\,q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2} {}^{\alpha}_{\alpha} \right|}. \tag{7.17}$$

Denote by

$$d_{il}^{\mathbf{A}} := \sum_{\beta \in J_{s_1,\, m}\{i\}} \mathrm{cdet}_i \left( (\mathbf{A}\mathbf{W}_1)^{k_1+2}_{.\,i} \left(\bar{\mathbf{d}}_{.\,l}\right) \right) {}^{\beta}_{\beta} = \sum_{t=1}^{n} \sum_{\beta \in J_{s_1,\, m}\{i\}} \mathrm{cdet}_i \left( (\mathbf{A}\mathbf{W}_1)^{k_1+2}_{.\,i} \left(\mathbf{e}_{.\,t}\right) \right) {}^{\beta}_{\beta} \bar{d}_{tl}$$

the $l$-th component of a row-vector $\mathbf{d}^{\mathbf{A}}_{i\,.} = (d^{\mathbf{A}}_{i1}, ..., d^{\mathbf{A}}_{iq})$ for all $l = \overline{1, q}$. Substituting it in (7.17), we have

$$x_{ij} = \frac{\sum\limits_{l=1}^{q} d_{il}^{\mathbf{A}} \sum\limits_{\alpha \in I_{s_2,q}\{j\}} \mathrm{rdet}_j \left( (\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.}(\mathbf{e}_{l.}) \right) {}^{\alpha}_{\alpha}}{\sum\limits_{\beta \in J_{s_1,\, m}} \left| (\mathbf{A}\mathbf{W}_1)^{k_1+2} {}^{\beta}_{\beta} \right| \sum\limits_{\alpha \in I_{s_2,\, q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2} {}^{\alpha}_{\alpha} \right|}.$$

Since $\sum\limits_{l=1}^{q} d_{il}^{\mathbf{A}} \mathbf{e}_{l.} = \mathbf{d}^{\mathbf{A}}_{i\,.}$, then it follows (7.8).

If we denote by

$$d_{tj}^{\mathbf{B}} := \sum_{l=1}^{q} \bar{d}_{tl} \sum_{\alpha \in I_{s_2,q}\{j\}} \mathrm{rdet}_j \left( (\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.}(\mathbf{e}_{l.}) \right)^{\alpha}_{\alpha} = \sum_{\alpha \in I_{s_2,q}\{j\}} \mathrm{rdet}_j \left( (\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.}(\bar{\mathbf{d}}_{t.}) \right)^{\alpha}_{\alpha}$$

the $t$-th component of a column-vector $\mathbf{d}^{\mathbf{B}}_{.\,j} = (d^{\mathbf{B}}_{1j}, ..., d^{\mathbf{B}}_{nj})^T$ for all $t = \overline{1, n}$ and substituting it in (7.17), we obtain

$$x_{ij} = \frac{\sum\limits_{t=1}^{n} \sum\limits_{\beta \in J_{s_1,\, m}\{i\}} \mathrm{cdet}_i \left( (\mathbf{A}\mathbf{W}_1)^{k_1+2}_{.\,i} \left(\mathbf{e}_{.\,t}\right) \right) {}^{\beta}_{\beta} \, d_{tj}^{\mathbf{B}}}{\sum\limits_{\beta \in J_{r_1,\, n}} \left| (\mathbf{A}^*\mathbf{A}) {}^{\beta}_{\beta} \right| \sum\limits_{\alpha \in I_{r_2,p}} \left| (\mathbf{B}\mathbf{B}^*)^{\alpha}_{\alpha} \right|}.$$

Since $\sum\limits_{t=1}^{n} \mathbf{e}_{.t} d_{tj}^{\mathbf{B}} = \mathbf{d}^{\mathbf{B}}_{.\,j}$, then it follows (7.7). □

**Remark 7.2.** *To establish the Cramer rule of (7.3) we shall not use the determinantal representations (6.30) and (6.30) for (7.5) because corresponding determinantal representations of it's solution will be too cumbersome. But they are suitable in the following corollaries.*

**Remark 7.3.** *In the complex case, i.e.* $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{W}_1 \in \mathbb{C}_{r_1}^{n\times m}$, $\mathbf{W}_2 \in \mathbb{C}_{r_2}^{q\times p}$, *and* $\mathbf{D} \in \mathbb{C}^{n\times p}$*, we can substitute usual determinants for all corresponding row and column determinants in (7.6), (7.7) and (7.7).*

*Because in the case ii), the conditions* $\mathbf{AW}_1 \in \mathbb{H}^{m\times m}$ *and* $\mathbf{W}_2\mathbf{B} \in \mathbb{H}^{q\times q}$ *be Hermitian are not necessary, then we have,*

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{s_1,\,m}\{i\}} \left|(\mathbf{AW}_1)^{k_1+2}_{\,.\,i}\left(\mathbf{d}^{\mathbf{B}}_{.\,j}\right)^{\beta}_{\beta}\right|}{\sum\limits_{\beta\in J_{s_1,\,m}} \left|(\mathbf{AW}_1)^{k_1+2}\,{}^{\beta}_{\beta}\right| \sum\limits_{\alpha\in I_{s_2,\,q}} \left|(\mathbf{W}_2\mathbf{B})^{k_2+2}\,{}^{\alpha}_{\alpha}\right|},$$

*or*

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{s_2,q}\{j\}} \left|(\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.}(\mathbf{d}^{\mathbf{A}}_{i\,.})\,{}^{\alpha}_{\alpha}\right|}{\sum\limits_{\beta\in J_{s_1,\,m}} \left|(\mathbf{AW}_1)^{k_1+2}\,{}^{\beta}_{\beta}\right| \sum\limits_{\alpha\in I_{s_2,\,q}} \left|(\mathbf{W}_2\mathbf{B})^{k_2+2}\,{}^{\alpha}_{\alpha}\right|},$$

*where*

$$\mathbf{d}^{\mathbf{B}}_{.\,j} = \left(\sum_{\alpha\in I_{s_2,q}\{j\}} \left|(\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\,.}(\bar{\mathbf{d}}_{t.})^{\alpha}_{\alpha}\right|\right) \in \mathbb{C}^{n\times 1}, \quad t = \overline{1,n}$$

$$\mathbf{d}^{\mathbf{A}}_{i\,.} = \left(\sum_{\beta\in J_{s_1,\,m}\{i\}} \left|(\mathbf{AW}_1)^{k_1+2}_{\,.\,i}\left(\bar{\mathbf{d}}_{.\,l}\right)^{\beta}_{\beta}\right|\right) \in \mathbb{C}^{1\times q}, \quad l = \overline{1,q}$$

*are the column vector and the row vector, respectively.* $\bar{\mathbf{d}}_{i.}$ *and* $\bar{\mathbf{d}}_{.j}$ *are the i-th row and the j-th column of* $\bar{\mathbf{D}}$ *for all* $i = \overline{1,n}$, $j = \overline{1,p}$. *These determinantal representations are most applicable for the complex case.*

**Corollary 7.1.** *Suppose the following restricted matrix equation is given,*

$$\mathbf{WAWX} = \mathbf{D}, \tag{7.18}$$

$$\mathcal{R}_r(\mathbf{X}) \subset \mathcal{R}_r\left((\mathbf{AW})^k\right),\ \mathcal{N}_l(\mathbf{X}) \supset \mathcal{N}_l\left((\mathbf{WA})^k\right), \tag{7.19}$$

*where* $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{W} \in \mathbb{H}_{r_1}^{n\times m}$ *with* $k = \max\{Ind(\mathbf{AW}), Ind\,(\mathbf{WA})\}$*, and* $\mathbf{D} \in \mathbb{H}^{n\times p}$ *. If* $\mathcal{R}_r(\mathbf{D}) \subset \mathcal{R}_r\left((\mathbf{AW})^k\right)$ *and* $\mathcal{N}_l(\mathbf{D}) \supset \mathcal{N}_l\left((\mathbf{WA})^k\right)$*, then the restricted matrix equation (7.18-7.19) has a unique solution,*

$$\mathbf{X} = \mathbf{A}_{d,W}\mathbf{D}, \tag{7.20}$$

*which possess the following determinantal representations for all* $i=\overline{1,m}$, $j=\overline{1,p}$,

*i)*

$$x_{ij}=\frac{\sum\limits_{t=1}^{n}\sum\limits_{\beta\in J_{r_1,\,m}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{W}^*\mathbf{W}\right)_{.t}\left(\hat{\mathbf{w}}_{.t}\right)\right)_{\beta}^{\beta}\sum\limits_{\beta\in J_{r,\,n}\{t\}}\mathrm{cdet}_t\left(\left(\left(\mathbf{U}^{2k+1}\right)^*\mathbf{U}^{2k+1}\right)_{.t}\left(\hat{\mathbf{d}}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,m}}\left|\left(\mathbf{W}^*\mathbf{W}\right)_{\beta}^{\beta}\right|\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\left(\mathbf{U}^{2k+1}\right)^*\mathbf{U}^{2k+1}\right)_{\beta}^{\beta}\right|}\tag{7.21}$$

*where* $\mathbf{U}=\mathbf{WA}$, $\hat{\mathbf{d}}_{.j}$ *is the* $j$*-th column of* $\hat{\mathbf{D}}=\hat{\mathbf{U}}\mathbf{D}=(\mathbf{U}^{2k+1})^*\mathbf{U}^k\mathbf{D}$, $\hat{\mathbf{W}}=\mathbf{W}^*\mathbf{U}^k$, *and* $r=\mathrm{rank}(\mathbf{WA})^{k+1}=\mathrm{rank}(\mathbf{WA})^k$.

*ii)*

$$x_{ij}=\sum_{q=1}^{m}(v_{iq}^{D})^{(2)}r_{qj},\tag{7.22}$$

*where* $(v_{iq}^{D})^{(2)}$ *can be obtained by (6.10) and* $\mathbf{AD}=\mathbf{R}=(r_{qj})\in\mathbb{H}^{m\times p}$.

*iii) If* $\mathbf{AW}\in\mathbb{H}^{m\times m}$ *is Hermitian, then*

$$x_{ij}=\frac{\sum\limits_{\beta\in J_{r,\,m}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{AW}\right)_{.i}^{k+2}\left(\mathbf{f}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,\,m}}\left|\left(\mathbf{AW}\right)^{k+2}{}_{\beta}^{\beta}\right|},\tag{7.23}$$

*where* $\mathbf{f}_{.j}$ *is the* $j$*-th column of* $\mathbf{F}=\bar{\mathbf{V}}\mathbf{D}=(\mathbf{AW})^k\mathbf{AD}$.

*Proof.* To derive a Cramer's rule (7.21), we use the determinantal representation (6.19) for $\mathbf{A}_{d,W}$. Then

$$x_{ij}=\sum_{s=1}^{p}a_{is}^{d,\mathbf{W}}d_{sj}=$$
$$\sum_{s=1}^{p}\left[\frac{\sum\limits_{t=1}^{n}\sum\limits_{\beta\in J_{r_1,\,m}\{i\}}\mathrm{cdet}_i(\mathbf{W}^*\mathbf{W})_{.t}\left(\hat{\mathbf{w}}_{.t}\right)_{\beta}^{\beta}\sum\limits_{\beta\in J_{r,\,n}\{t\}}\mathrm{cdet}_t\left(\left(\mathbf{U}^{2k+1}\right)^*\mathbf{U}^{2k+1}\right)_{.t}\left(\hat{\mathbf{u}}_{.s}\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_1,\,m}}\left|\left(\mathbf{W}^*\mathbf{W}\right)_{\beta}^{\beta}\right|\sum\limits_{\beta\in J_{r,\,n}}\left|\left(\left(\mathbf{U}^{2k+1}\right)^*\mathbf{U}^{2k+1}\right)_{\beta}^{\beta}\right|}\right]d_{sj}\tag{7.24}$$

Denote $\hat{\mathbf{D}}=\hat{\mathbf{U}}\mathbf{D}=(\mathbf{U}^{2k+1})^*\mathbf{U}^k\mathbf{D}$, where $\hat{\mathbf{D}}=\left(\hat{d}_{sj}\right)\in\mathbb{H}^{n\times p}$. Since

$$\sum_{s=1}^{p}\hat{\mathbf{u}}_{.s}d_{sj}=\hat{\mathbf{d}}_{.j},$$

where $\hat{\mathbf{d}}_{.j}$ is the $j$-th column of $\hat{\mathbf{D}}$, then (7.21) follows from (7.24).

Cramer's rules (7.22) and (7.23) immediately follow from Theorem 7.1 by putting $\mathbf{W}_1 = \mathbf{W}$, $\mathbf{W}_2\mathbf{B} = \mathbf{I}$. □

**Remark 7.4.** *In the complex case, i.e.* $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{W} \in \mathbb{C}^{n\times m}_{r_1}$, *and* $\mathbf{D} \in \mathbb{C}^{n\times p}$, *we substitute usual determinants for all corresponding row and column determinants in (7.21), (7.22), and (7.23).*

*Note that in the case iii), the condition* $\mathbf{AW} \in \mathbb{C}^{m\times m}$ *be Hermitian is not necessary, then in the complex case (7.23) will have the form*

$$x_{ij} = \frac{\sum\limits_{\beta\in J_{r,\,m}\{i\}} \left|\left((\mathbf{AW})^{k+2}_{.\,i}\left(\mathbf{f}_{.j}\right)\right) {}^{\beta}_{\beta}\right|}{\sum\limits_{\beta\in J_{r,\,m}} \left|(\mathbf{AW})^{k+2}\, {}^{\beta}_{\beta}\right|},$$

*where* $\mathbf{f}_{.j}$ *is the* $j$*-th column of* $\mathbf{F} = \bar{\mathbf{V}}\mathbf{D} = (\mathbf{AW})^k\mathbf{AD}$.

**Corollary 7.2.** *Suppose the following restricted matrix equation is given,*

$$\mathbf{XWBW} = \mathbf{D}, \tag{7.25}$$

$$\mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l\left((\mathbf{BW})^k\right),\ \mathcal{N}_r(\mathbf{X}) \supset \mathcal{N}_r\left((\mathbf{BA})^k\right), \tag{7.26}$$

*where* $\mathbf{B} \in \mathbb{H}^{p\times q}$, $\mathbf{W} \in \mathbb{H}^{q\times p}_{r_1}$ *with* $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WB})\}$, *and* $\mathbf{D} \in \mathbb{H}^{n\times p}$. *If* $\mathcal{R}_l(\mathbf{D}) \subset \mathcal{R}_l\left((\mathbf{BW})^k\right)$ *and* $\mathcal{N}_r(\mathbf{D}) \supset \mathcal{N}_r\left((\mathbf{WB})^k\right)$, *then the restricted matrix equation (7.25-7.26) has a unique solution,*

$$\mathbf{X} = \mathbf{DB}_{d,W}, \tag{7.27}$$

*which possess the following determinantal representations for* $i = \overline{1,n}$, $j = \overline{1,q}$,

*i)*

$$x_{ij} = \frac{\sum\limits_{l=1}^{p}\sum\limits_{\alpha\in I_{r,\,p}\{l\}} \mathrm{rdet}_l\left(\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^*\right)_{l.}\left(\check{\mathbf{d}}_{i\,.}\right)\right){}^{\alpha}_{\alpha} \sum\limits_{\alpha\in I_{r_1,q}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{WW}^*\right)_{j\,.}\left(\check{\mathbf{w}}_{l.}\right)\right){}^{\alpha}_{\alpha}}{\sum\limits_{\alpha\in I_{r,\,m}} \left|\left(\mathbf{V}^{2k+1}\left(\mathbf{V}^{2k+1}\right)^*\right){}^{\alpha}_{\alpha}\right| \sum\limits_{\alpha\in I_{r_1,\,n}} \left|\left(\mathbf{WW}^*\right){}^{\alpha}_{\alpha}\right|} \tag{7.28}$$

*where* $\mathbf{V} = \mathbf{B}\mathbf{W}$, $\check{\mathbf{d}}_{i.}$ *is the* $i$*-th row of* $\check{\mathbf{D}} = \mathbf{D}\check{\mathbf{V}} = \mathbf{D}\mathbf{V}^k(\mathbf{V}^{2k+1})^*$, $\check{\mathbf{w}}_{l.}$ *is the* $l$*-th row of* $\check{\mathbf{W}} = \mathbf{V}^k\mathbf{W}^*$, *and* $r = \operatorname{rank}(\mathbf{B}\mathbf{W})^{k+1} = \operatorname{rank}(\mathbf{B}\mathbf{W})^k$.

*ii)*

$$x_{ij} = \sum_{t=1}^{q} l_{it}(u_{tj}^D)^{(2)}, \tag{7.29}$$

*where* $(u_{tj}^D)^{(2)}$ *can be obtained by (6.8) and* $\mathbf{D}\mathbf{B} = \mathbf{L} = (l_{it}) \in \mathbb{H}^{n\times q}$.

*iii) If* $\mathbf{W}\mathbf{B} \in \mathbb{H}^{q\times q}$ *is Hermitian, then*

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{r,q}\{j\}} \operatorname{rdet}_j\left((\mathbf{W}\mathbf{B})_{j\,.}^{k+2}(\mathbf{g}_{i.})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,\,q}} \left|(\mathbf{W}\mathbf{B})^{k+2}{}_{\alpha}^{\alpha}\right|}. \tag{7.30}$$

*where* $\mathbf{g}_{i.}$ *is the* $i$*-th row of* $\mathbf{G} = \mathbf{D}\mathbf{B}(\mathbf{W}\mathbf{B})^k$ *for all* $i = \overline{1,n}$.

*Proof.* The proof is similar to the proof of Corollary 7.1 in the point i), and follows from Theorem 7.1 by putting $\mathbf{W}_2 = \mathbf{W}$, $\mathbf{A}\mathbf{W}_1 = \mathbf{I}$. $\square$

**Remark 7.5.** *In the complex case, i.e.* $\mathbf{B} \in \mathbb{C}^{p\times q}$, $\mathbf{W} \in \mathbb{C}_{r_1}^{q\times p}$, *and* $\mathbf{D} \in \mathbb{C}^{n\times p}$, *we substitute usual determinants for all corresponding row and column determinants in (7.28), (7.29), and (7.30). Herein the condition* $\mathbf{W}\mathbf{B} \in \mathbb{C}^{n\times n}$ *be Hermitian is not necessary, then in the complex case (7.30) can be represented as follows,*

$$x_{ij} = \frac{\sum\limits_{\alpha\in I_{r,q}\{j\}} \left|\left((\mathbf{W}\mathbf{B})_{j\,.}^{k+2}(\mathbf{g}_{i.})\right)_{\alpha}^{\alpha}\right|}{\sum\limits_{\alpha\in I_{r,\,q}} \left|(\mathbf{W}\mathbf{B})^{k+2}{}_{\alpha}^{\alpha}\right|}.$$

*where* $\mathbf{g}_{i.}$ *is the* $i$*-th row of* $\mathbf{G} = \mathbf{D}\mathbf{B}(\mathbf{W}\mathbf{B})^k$ *for all* $i = \overline{1,n}$.

## 7.3. Examples

1. Let us consider the matrix equation

$$\mathbf{W}\mathbf{A}\mathbf{W}\mathbf{X} = \mathbf{D} \tag{7.31}$$

with the restricted conditions (7.19), where $\mathbf{W}$ and $\mathbf{A}$ are the same as in Example 64., and

$$\mathbf{D} = \begin{pmatrix} k & i \\ i & -j \\ 1 & -i \end{pmatrix}.$$

Therefore, the matrices $\mathbf{V} = \mathbf{AW}$, $\mathbf{U} = \mathbf{WA}$, $\left(\mathbf{U}^5\right)^* \mathbf{U}^5$, $\mathbf{W}^*$, $\mathbf{W}^*\mathbf{W}$, $\hat{\mathbf{W}} = \mathbf{W}^*\mathbf{U}^2$ are the same that in Example 64. as well, and

$$\hat{\mathbf{D}} = (\mathbf{U}^5)^*\mathbf{U}^2\mathbf{D} = \begin{pmatrix} i-j-k & -j \\ 1+3i+6j-2k & 4i-2k \\ 0 & 0 \end{pmatrix}.$$

So, by (7.21)

$$x_{11} = \frac{\sum\limits_{t=1}^{3} \sum\limits_{\beta\in I_{3,\,4}\{1\}} \operatorname{cdet}_1 \left(\left(\mathbf{W}^*\mathbf{W}\right)_{.1} \left(\hat{\mathbf{w}}_{.t}\right)\right) {}_{\beta}^{\beta} \sum\limits_{\beta\in J_{2,\,3}\{t\}} \operatorname{cdet}_t \left(\left(\left(\mathbf{U}^5\right)^* \mathbf{U}^5\right)_{.t} \left(\hat{\mathbf{d}}_{.1}\right)\right) {}_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{3,\,4}} \left|\left(\mathbf{W}^*\mathbf{W}\right) {}_{\beta}^{\beta}\right| \sum\limits_{\beta\in J_{2,\,3}} \left|\left(\left(\mathbf{U}^5\right)^* \mathbf{U}^5\right) {}_{\beta}^{\beta}\right|},$$

where

$$\sum_{\beta\in I_{3,\,4}\{1\}} \operatorname{cdet}_1 \left(\left(\mathbf{W}^*\mathbf{W}\right)_{.1} \left(\hat{\mathbf{w}}_{.1}\right)\right) {}_{\beta}^{\beta} =$$

$$\operatorname{cdet}_1 \begin{pmatrix} k & i & -j \\ 0 & 2 & 0 \\ i & 0 & 1 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} k & i & j \\ 0 & 2 & -2k \\ 0 & 2k & 1 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} k & -j & j \\ i & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 0,$$

$$\sum_{\beta\in I_{3,\,4}\{1\}} \operatorname{cdet}_1 \left(\left(\mathbf{W}^*\mathbf{W}\right)_{.1} \left(\hat{\mathbf{w}}_{.2}\right)\right) {}_{\beta}^{\beta} = -2j,$$

$$\sum_{\beta\in I_{3,\,4}\{1\}} \operatorname{cdet}_1 \left(\left(\mathbf{W}^*\mathbf{W}\right)_{.1} \left(\hat{\mathbf{w}}_{.3}\right)\right) {}_{\beta}^{\beta} = 0, \sum_{\beta\in J_{3,\,4}} \left|\left(\mathbf{W}^*\mathbf{W}\right) {}_{\beta}^{\beta}\right| = 2,$$

and

$$\sum_{\beta\in J_{2,\,3}\{1\}} \operatorname{cdet}_1 \left(\left(\left(\mathbf{U}^5\right)^* \mathbf{U}^5\right)_{.1} \left(\hat{\mathbf{d}}_{.1}\right)\right) {}_{\beta}^{\beta} =$$

$$\operatorname{cdet}_1 \begin{pmatrix} i-j-k & -2i-3k \\ 1+3i+6j-2k & 14 \end{pmatrix} + \operatorname{cdet}_1 \begin{pmatrix} i-j-k & 0 \\ 0 & 0 \end{pmatrix} = -2i-j-k,$$

$$\sum_{\beta\in J_{2,3}\{2\}} \operatorname{cdet}_2\left(\left(\left(\mathbf{U}^5\right)^*\mathbf{U}^5\right)_{.2}(\hat{\mathbf{d}}_{.1})\right)_{\beta}^{\beta}=j,$$

$$\sum_{\beta\in J_{2,3}\{3\}} \operatorname{cdet}_3\left(\left(\left(\mathbf{U}^5\right)^*\mathbf{U}^5\right)_{.3}(\hat{\mathbf{d}}_{.1})\right)_{\beta}^{\beta}=0,\ \sum_{\beta\in J_{2,3}}\left|\left(\left(\mathbf{U}^5\right)^*\mathbf{U}^5\right)_{\beta}^{\beta}\right|=1.$$

Therefore,

$$x_{11}=\frac{0\cdot(-2i-j-k)+(-2j)\cdot j+0\cdot 0}{2\cdot 1}=1,$$
$$x_{12}=\frac{0\cdot(-2+2j)+(-2j)\cdot i+0\cdot 0}{2\cdot 1}=k,$$
$$x_{21}=\frac{2j\cdot(-2i-j-k)+(10i-4k)\cdot j+0\cdot 0}{2\cdot 1}=1+i+7k,$$

$$x_{22}=\frac{2j\cdot(-2+2j)+(10i-4k)\cdot i+0\cdot 0}{2\cdot 1}=-7-4j,$$
$$x_{31}=\frac{10i\cdot(-2i-j-k)+j\cdot j+0\cdot 0}{2\cdot 1}=9.5+5j-5k,$$
$$x_{32}=\frac{10i\cdot(-2+2j)+j\cdot i+0\cdot 0}{2\cdot 1}=-10i+9.5k,$$

We finally get,

$$\mathbf{X}=\begin{pmatrix}1 & k\\ 1+i+7k & -7-4j\\ 9.5+5j-5k & -10i+9.5k\end{pmatrix}.$$

2. Let now we consider the matrix equation

$$\mathbf{W}_1\mathbf{A}\mathbf{W}_1\mathbf{X}\mathbf{W}_2\mathbf{B}\mathbf{W}_2=\mathbf{D}, \tag{7.32}$$

with the constraints (7.4), where

$$\mathbf{A}=\begin{pmatrix}k & 0 & i & 0\\ -j & k & 0 & 1\\ 0 & 1 & 0 & -k\end{pmatrix},\ \mathbf{W}_1=\begin{pmatrix}k & -j & 0\\ 0 & k & 1\\ i & 0 & 0\\ 0 & 1 & -k\end{pmatrix},\mathbf{W}_2=\begin{pmatrix}k & -i\\ j & 0\\ 0 & 1\end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} k & j & 0 \\ j & 0 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} i & -1 \\ k & 0 \\ 0 & j \\ -1 & 0 \end{pmatrix}.$$

Since the following matrices are Hermitian

$$\mathbf{V} = \mathbf{A}\mathbf{W}_1 = \begin{pmatrix} -2 & i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbf{U} = \mathbf{W}_2\mathbf{B} = \begin{pmatrix} 0 & -i & -i \\ i & -1 & 0 \\ i & 0 & -1 \end{pmatrix},$$

then we can find the W-weighted Drazin inverse solution of (7.32) by its determinantal representation (7.7). We have

$$k_1 = \max\{Ind(\mathbf{A}\mathbf{W}_1), Ind\,(\mathbf{W}_1\mathbf{A})\} = 1,$$
$$k_2 = \max\{Ind(\mathbf{B}\mathbf{W}_2), Ind\,(\mathbf{W}_2\mathbf{B})\} = 1,$$

and $s_1 = \operatorname{rank}(\mathbf{A}\mathbf{W}_1) = 2$, $s_2 = \operatorname{rank}(\mathbf{W}_2\mathbf{B}) = 2$. Since

$$(\mathbf{A}\mathbf{W}_1)^3 = \begin{pmatrix} -13 & 8i & 0 \\ -8i & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix}, (\mathbf{W}_2\mathbf{B})^3 = \begin{pmatrix} 0 & -3i & -3i \\ 3i & -3 & 0 \\ 3i & 0 & 3 \end{pmatrix},$$

then

$$\sum_{\beta \in J_{2,3}} \left| (\mathbf{A}\mathbf{W}_1)^3 \, {}^{\beta}_{\beta} \right| = 1, \sum_{\alpha \in I_{2,3}} \left| (\mathbf{W}_2\mathbf{B})^3 \, {}^{\alpha}_{\alpha} \right| = -27.$$

Therefore,

$$\bar{\mathbf{D}} = \mathbf{A}\mathbf{W}_1\mathbf{A}\mathbf{D}\mathbf{B}\mathbf{W}_2\mathbf{B} = \begin{pmatrix} 2i + j & -7 + k & -5 + 2k \\ -1 + k & -5i - j & -4i - 2j \\ 0 & 0 & 0 \end{pmatrix}.$$

By (7.9), we can get

$$\mathbf{d}^{\mathbf{B}}_{.1} = \begin{pmatrix} 36i - 9j \\ -27 - 9k \\ 0 \end{pmatrix}, \quad \mathbf{d}^{\mathbf{B}}_{.2} = \begin{pmatrix} -27 \\ -18i \\ 0 \end{pmatrix}, \quad \mathbf{d}^{\mathbf{B}}_{.3} = \begin{pmatrix} 9 - 9k \\ 9i + 3j \\ 0 \end{pmatrix}.$$

Since

$$(\mathbf{A}\mathbf{W}_1)^3_{.1}\left(\mathbf{d}^{\mathbf{B}}_{.1}\right) = \begin{pmatrix} 36i - 9j & 8i & 0 \\ -27 - 9k & -5 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

then finally we obtain

$$x_{11} = \frac{\sum\limits_{\beta \in J_{2,3}\{1\}} \operatorname{cdet}_1 \left( (\mathbf{AW}_1)^3_{.1} \left( \mathbf{d}^{\mathbf{B}}_{.1} \right) \right) {}^{\beta}_{\beta}}{\sum\limits_{\beta \in J_{2,3}} \left| (\mathbf{AW}_1)^3 {}^{\beta}_{\beta} \right| \sum\limits_{\alpha \in I_{2,3}} \left| (\mathbf{W}_2\mathbf{B})^3 {}^{\alpha}_{\alpha} \right|} = \frac{36i - 27j}{-27} = \frac{-4i + 3j}{3},$$

Similarly,

$$x_{12} = \frac{\operatorname{cdet}_1 \begin{pmatrix} -27 & 8i \\ -18i & -5 \end{pmatrix}}{-27} = \frac{1}{3}, \ x_{13} = \frac{\operatorname{cdet}_1 \begin{pmatrix} 9 - 9k & 8i \\ 9i - 3j & -5 \end{pmatrix}}{-27} = \frac{-9 - 7k}{9},$$

$$x_{21} = \frac{\operatorname{cdet}_2 \begin{pmatrix} -13 & 36i - 9j \\ -8i & -27 - 9k \end{pmatrix}}{-27} = \frac{-7 - 5k}{3}, \ x_{22} = \frac{\operatorname{cdet}_2 \begin{pmatrix} -13 & -27 \\ -8i & -18i \end{pmatrix}}{-27} = \frac{-2i}{3},$$

$$x_{23} = \frac{\operatorname{cdet}_2 \begin{pmatrix} -13 & -9 - 9k \\ -8i & 9i + 3j \end{pmatrix}}{-27} = \frac{15i - 11j}{9}, \ x_{31} = x_{32} = x_{33} = 0.$$

So, the W-weighted Drazin inverse solution of (7.32) are

$$\mathbf{X} = \frac{1}{9} \begin{pmatrix} -12i + 9j & 3 & -9 - 7k \\ -21 - 15k & -6i & 15i - 11j \\ 0 & 0 & 0 \end{pmatrix}.$$

## Conclusion

In this chapter, we have obtained determinantal representations of the Drazin and W-weighted Drazin inverses over the quaternion skew field. We have derived determinantal representations of the Drazin inverse for both Hermitian and arbitrary matrices over the quaternion skew field by the theory of column-row determinants recently introduced by the author. Using obtained determinantal representations of the Drazin inverse we have get explicit representation formulas (analogs of Cramer's rule) for the Drazin inverse solutions of the quaternionic matrix equations $\mathbf{AXB} = \mathbf{D}$ and, consequently, $\mathbf{AX} = \mathbf{D}$, $\mathbf{XB} = \mathbf{D}$ in both cases when $\mathbf{A}$ and $\mathbf{B}$ are Hermitian and arbitrary. We also have obtain determinantal representations of solutions of the differential quaternion-matrix equations, $\mathbf{X}' + \mathbf{AX} = \mathbf{B}$ and $\mathbf{X}' + \mathbf{XA} = \mathbf{B}$, where $\mathbf{A}$ is noninvertible.

Also, we have obtained new determinantal representations of the W-weighted Drazin inverse over the quaternion skew field. We have gave determinantal representations of the W-weighted Drazin inverse by using previously obtained determinantal representations of the Drazin inverse, the Moore-Penrose inverse, and the limit representations of the W-weighted Drazin inverse in some special case. Using these determinantal representations of the W-weighted Drazin inverse, explicit formulas for determinantal representations of the W-weighted Drazin inverse solutions of the quaternionic matrix equations $\mathbf{WAWX} = \mathbf{D}$, $\mathbf{XWAW} = \mathbf{D}$, and $\mathbf{W}_1\mathbf{AW}_1\mathbf{XW}_2\mathbf{BW}_2 = \mathbf{D}$ have been obtained.

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# INDEX

## D



## E



## F

## N



## O



## P



## Q

## R



## S



## T

## U



## V



## W