also remains valid in the spaces of analytic functions introduced in Ref. 2.

The function $\varphi = G^{(m)} + \Pi_1 P_0 f$ is a solution of Eq. (8) for $g = f_0$ and $h = h_1 + h_2 + h_3$, where

$$h_1 = \epsilon^m (Lg^{(m)} + 2\Gamma(f_0, g^{(m)})),$$

$$h_2 = \sum_{j=m+1}^{2(m-1)} \epsilon^j \sum_{l=j+1-m}^{m-1} \Gamma(g^{(l)}, g^{(j-l)}),$$

$$h_3 = \epsilon \Pi_1 P_0 D(G^{(m)} - P_1 f).$$

Consequently, the function $P_1 f$ is consistent with the representation

$$P_1 f = G^{(m)} + \Phi^{(m)} + R^{(m)},$$

$$\Phi^{(m)} = V(f_1(0), f_0, 0) - V(G^{(m)}(0), f_0, 0),$$

$$R^{(m)} = V(G^{(m)}(0), f_0, 0) - V(G^{(m)}(0), f_0, h).$$

It is easy to derive Theorem 2 from these relations and Theorem 1.

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Lagrangian analysis of the third-order invariant equations of motion in the relativistic mechanics of classical particles

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Interest in the motion of classical scalar and spinning particles in gauge and gravitational fields has recently increased. The corresponding equations are generalizations of the Lorentz equations and the Papapetrou–Pirani equations.[1,2] At the same time, the Mathiesson–Papapetrou equations with the Pirani condition $u_\beta S^{\alpha\beta} = 0$ ($u^\alpha = \dot{x}^\alpha$) yield a third-order equation of motion[3]

$$m_0 \frac{Du^\alpha}{ds} = S^{\alpha\beta} \frac{D^2 u_\beta}{ds^2} - \frac{1}{2} R^{\alpha}_{\ \beta\gamma\delta} u^\beta S^{\gamma\delta},$$
$$u_\beta u^\beta = 1. \tag{1}$$

The Mathiesson–Papapetrou equations and the preceding equations of Frenkel'[4] have been derived from a variational principle by many authors. The question of the existence of a Lagrangian for the Mathiesson equation (1), however, has so far not been investigated. On the other hand, if the particle is not a test particle, the radiational friction force that is in the equation of motion contains a third-order term[5] as, e.g., in the Dirac–Lorentz equation

$$m_0 \left[\frac{\dot{u}^\alpha}{\|u\|^2} - \frac{\dot{u}_\beta u^\beta}{\|u\|^4} u^\alpha \right] = \frac{e}{\|u\|} H^{\alpha\beta} u_\beta$$
$$+ \frac{2e^2}{3} \left[\frac{\ddot{u}^\alpha}{\|u\|^3} - \frac{\ddot{u}_\beta u^\beta}{\|u\|^5} u^\alpha - 3 \frac{\dot{u}_\beta u^\beta}{\|u\|^5} \dot{u}^\alpha + 3 \frac{(\dot{u}_\beta u^\beta)^2}{\|u\|^7} u^\alpha \right]. \tag{2}$$

The question of the existence of a Lagrangian for the Dirac–Lorentz equation so far has not been solved fully.[6] Equation (2) is invariant with respect to a change in the parametrization of the integral curve. If we set $m_0 = 0$ and $H^{\alpha\beta} = 0$, the equation becomes the equation for the geodesic circles, which determines the motion of relativistic particles and which was investigated in connection with the problem of a uniformly accelerated coordinate system.[7]

The parametrically invariant form of Mathiesson equations (1) can be obtained immediately from Dixon's equations[8]

$$\frac{DP^\alpha}{d\lambda} = \frac{1}{2} R^{\alpha}_{\ \beta\gamma\delta} u^\beta S^{\gamma\delta}, \quad \frac{DS^{\alpha\beta}}{d\lambda} = 2 P^{[\alpha} u^{\beta]},$$

where λ is an arbitrary parameter along the world line. Here we propose an equivalent formulation in terms of the spin four-vector

$$\|u\| \sigma_\alpha = \frac{\sqrt{|g|}}{2} \epsilon_{\alpha\beta\gamma\delta} u^\beta S^{\gamma\delta};$$

$$m_0 \|u\| \left[\|u\|^4 \frac{Du_\alpha}{d\lambda} - \|u\|^2 u_\beta \frac{Du^\beta}{d\lambda} u_\alpha \right] \tag{3}$$

$$= \|u\| \sigma^\nu \left[\sqrt{|g|} \|u\|^2 \epsilon_{\alpha\beta\gamma\nu} \frac{D^2 u^\beta}{d\lambda} u^\gamma - 3\sqrt{|g|} u_\delta \frac{Du^\delta}{d\lambda} \epsilon_{\alpha\beta\gamma\nu} \frac{Du^\beta}{d\lambda} u^\gamma \right.$$
$$\left. - \frac{\sqrt{|g|}}{2} \|u\|^4 \epsilon_{\gamma\delta\mu\nu} R_{\alpha\beta}^{\ \ \gamma\delta} u^\beta u^\mu \right]. \tag{4}$$

It is necessary to supplement Eq. (4) with the condition $\|u\| u_\beta \sigma^\beta = 0$ and with the spin part of Dixon's equations

$$\|u\|^3 \frac{D\sigma^\alpha}{d\lambda} + \|u\| \sigma_\beta \frac{Du^\beta}{d\lambda} u^\alpha = 0. \tag{5}$$

If the world line of the particle is a null geodesic line, the expression $\|u\|\sigma_\alpha$ in Eqs. (4) and (5) should be understood in the sense of Eq. (3). If the particle has no rest mass, we should set $m_0 \|u\| = 0$ in Eq. (4).[9] For the motion of a particle with nonzero rest mass in a flat space-time the spin four-vector remains constant and Eq. (5) can be omitted.

In this communication we examine the question of the existence of the invariant third-order Euler–Poisson equations[10] in a pseudo-Euclidean space. Locally, the variational problem is determined by a Lagrangian density $L(t, \mathbf{x}, \mathbf{v}, \mathbf{v}')dt$ in the space of second-order jets $J_2(\mathbf{R}, \mathbf{R}^{n-1})$. Let p: $T_r\mathbf{R}^n \to J_r(\mathbf{R}, \mathbf{R}^{n-1})$ be the local expression

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of the canonical projection of the space of 1^r velocities $T_r\mathbf{R}^n = J_r(\mathbf{R}, \mathbf{R}^n)(0)$ onto the manifold $C_r(\mathbf{R}^n, 1)$ of one-dimensional contact elements in $\mathbf{R}^n$. Identifying $\mathbf{R}^n$ with the direct product $\mathbf{R}\times\mathbf{R}^{n-1}$, we denote the canonical coordinates in $T_4\mathbf{R}^n$ by $x = (t, \mathbf{x})$; $u = (u^0, \mathbf{u}) = \dot{x}$; $\dot{u} = \ddot{x}$; $\dddot{x}$; $\ddot{u}$. In the space $\mathbf{R}^n$, a parametrically invariant variational problem arises with a Lagrangian $\mathcal{L}(x, u, \dot{u}) = u^0 L(t, \mathbf{x}, \mathbf{v}\circ p, \mathbf{v}'\circ p)$, that is defined locally in $T_2\mathbf{R}^n$. We denote by $\mathbf{E}$ and $\mathcal{E} = (\mathcal{E}_0, \underline{\mathcal{E}})$ the Euler–Poisson expressions generated by the corresponding Lagrange functions L and $\mathcal{L}$. Then $u^0\mathcal{E}_0 + \mathbf{u}\cdot\underline{\mathcal{E}} = 0$ and the relation $\underline{\mathcal{E}}(x, u, \dot{u}, \ddot{u}) = u^0\mathbf{E}(t, \mathbf{x}, \mathbf{v}\circ p, \mathbf{v}'\circ p, \mathbf{v}''\circ p)$. holds. The arbitrary expression $\mathbf{E}(t, \mathbf{x}, \mathbf{v}', \mathbf{v}'')$ is then the Euler–Lagrange expression if and only if

$$\mathbf{E} = \mathbf{A}\cdot\mathbf{v}'' + (\mathbf{v}'\cdot\underline{\partial}_{\mathbf{v}})\mathbf{A}\cdot\mathbf{v}' + \mathbf{B}\cdot\mathbf{v}' + \mathbf{c}, \qquad (6)$$

where the skew-symmetric matrix $\mathbf{A}$, the matrix $\mathbf{B}$, and the column vector $\mathbf{c}$ depend only on the variables t, $\mathbf{x}$, and $\mathbf{v}$, and the following conditions are satisfied[11]:

$$\begin{aligned}
&\partial_{v[i}A_{jk]} = 0, \quad 2B_{[ij]} - 3D_{\mathbf{x}}A_{ij} = 0,\\
&2\partial_{v[i}B_{j]k} - 4\partial_{x[i}A_{j]k} + \partial_{xk}A_{ij} + 2D_{\mathbf{x}}\partial_{vk}A_{ij} = 0,\\
&\partial_{v(i}c_{j)} - D_{\mathbf{x}}B_{(ij)} = 0,\\
&2\partial_{vk}\partial_{v[i}c_{j]} - 4\partial_{x[i}B_{j]k} + D_{\mathbf{x}}^2\partial_{vk}A_{ij} + 6D_{\mathbf{x}}\partial_{x[i}A_{jk]} = 0,\\
&4\partial_{x[i}c_{j]} - 2D_{\mathbf{x}}\partial_{v[i}c_{j]} - D_{\mathbf{x}}^3A = 0.
\end{aligned} \qquad (7)$$

The symbols $D_{\mathbf{x}}$ and $D_{\mathbf{v}}$ denote the truncated total differential operators $D_{\mathbf{x}} = \partial_t + \mathbf{v}\cdot\partial_{\mathbf{x}}$, $D_{\mathbf{v}} = D_{\mathbf{x}} + \mathbf{v}'\cdot\underline{\partial}_{\mathbf{v}}$. The Euler morphism[12] $\mathbf{E}$: $J_3(\mathbf{R}, \mathbf{R}^{n-1}) \to T^*\mathbf{R} \otimes_{\mathbf{R}^n} T^*\mathbf{R}^{n-1}$, $\mathbf{E}(t, \mathbf{x}, \mathbf{v}, \mathbf{v}', \mathbf{v}'') = dt \otimes \mathbf{E}(t, \mathbf{x}, \mathbf{v}, \mathbf{v}', \mathbf{v}'')$, is an affine mapping over $J_2(\mathbf{R}^n, \mathbf{R}^{n-1})$. A Pfaffian form is therefore associated with it with values in the vector space $\mathbf{R}^{n-1*}$

$$\epsilon = \mathbf{A}\cdot d\mathbf{v}' + \kappa\, dt, \quad \kappa = (\mathbf{v}'\cdot\underline{\partial}_{\mathbf{v}})\mathbf{A}\cdot\mathbf{v}' + \mathbf{B}\cdot\mathbf{v}' + \mathbf{c}. \qquad (8)$$

The form ϵ and the contact $T\mathbf{R}^{n-1}$ valued form $\theta = (dx^i - v^i dt)\otimes\partial_{x^i} + (dv^i - v'^i dt)\otimes\partial_{v^i}$ generate a module $\mathfrak{M}(\epsilon, \theta)$ over the algebra of differential forms in the space $J_2(\mathbf{R}, \mathbf{R}^{n-1})$ with values in the vector space $\mathrm{End}(\mathbf{R}^{n-1*} \otimes T\mathbf{R}^{n-1})$. Let the pseudogroup of transformations Γ generated by the vector fields $X = \tau\partial_t + \xi\cdot\underline{\partial}_{\mathbf{x}}$, and $X_r = X + \xi^{(1)}\cdot\underline{\partial}_{\mathbf{v}} + \xi^{(2)}\cdot\underline{\partial}_{\mathbf{v}'} + \ldots + \xi^{(r)}\cdot\underline{\partial}_{\mathbf{v}(r-1)}$ [the continuation of the generator X into the space $J_r(\mathbf{R}, \mathbf{R}^{n-1})$] act in $\mathbf{R}^n$. The infinitesimal condition for the invariance of the Euler–Poisson equations with respect to the pseudogroup Γ is included in the invariance of the module $\mathfrak{M}(\epsilon, \theta)$ under the action of the vector field X_2. The definitive equations are

$$X_1(\mathbf{A}) = \Lambda\cdot\mathbf{A} - \mathbf{A}\cdot\xi^{(2)}_{\mathbf{v}'}, \quad X_2(\kappa) = \Lambda\cdot\kappa - \mathbf{A}\cdot D_{\mathbf{v}}\xi^{(2)} - \kappa D_{\mathbf{x}}\tau. \qquad (9)$$

The matrix Λ serves as an undetermined multiplier. The invariance below is assumed to be with respect to the pseudo-orthogonal group of transformations.

Let the Lagrangian $\mathcal{L}$ be translationally invariant. By solving partial differential equations (7) and (9) we can prove the following assertions.

1. In three-dimensional pseudo-Euclidean space there exists only a one-parameter family of invariant third-order Euler–Poisson equations:

$$\frac{m}{\|u\|^3}[\|u\|^2\dot{u} - (\dot{u}\cdot u)u] - \frac{\ddot{u}\times u}{\|u\|^3} + 3\frac{(\dot{u}\cdot u)}{\|u\|^5}\dot{u}\times u = 0. \qquad (10)$$

If $m = 0$, Eq. (10) is algebraically equivalent to the equation for the geodesic circles.

2. There are no third-order invariant Euler–Poisson equations in a four-dimensional pseudo-Euclidean space. However, in this case it is possible to find an invariant family of Euler–Poisson equations that depend on the four-vector parameter $\sigma = (\sigma^0, \boldsymbol{\sigma})$

$$\frac{m}{\|\sigma\|^3}\left[\frac{\dot{u}}{\|u\|} - \frac{(\dot{u}\cdot u)}{\|u\|^3}u\right] + \frac{*(\ddot{u}\wedge u\wedge\sigma)}{\|\sigma\wedge u\|^3} - 3\frac{(\sigma\wedge\dot{u})\cdot(\sigma\wedge u)}{\|\sigma\wedge u\|^5}*(\dot{u}\wedge u\wedge\sigma) = 0. \qquad (11)$$

A comparison with Eq. (4) shows that Eq. (11) describes the motion of free particles with a rest mass $m_0 = m\left[1 - \frac{(\sigma\cdot u)^2}{(\sigma\cdot\sigma)(u\cdot u)}\right]^{3/2}$ and a constant spin four-vector σ. If $m = 0$, Eq. (11), with the additional condition $\sigma\cdot u = 0$, describes the motion of massless timelike particles with a constant spin four-vector.

3. If $m = 0$, the set of integral lines in Eq. (11) includes those geodesic circles for which the unit four-velocity vector forms a constant arbitrary angle with the direction chosen by σ: $(\sigma\cdot u/\|u\|)^{\cdot} = 0$. The geodesic circles are separated from the set of integral lines in Eq. (11) by the conditions $m = 0$ and $(\sigma\cdot u)\dot{u}\wedge u = 0$.

4. Equation (10) also has a physical meaning. It describes the planar motion of a free particle with a rest mass of $m\|\sigma\|$ and a spin σ that is orthogonal to the plane of motion. These motions were considered in Ref. 13.

Using condition (7), we see that there are no Euler–Poisson equations that are equivalent to the Dirac–Lorentz equation [Eq. (2)].

The noninvariant Lagrangian for Eq. (10) has the following general form:

$$\mathcal{L} = \frac{1}{2\|u\|}\left[\frac{u_2(\dot{u}_1u_0 - \dot{u}_0u_1)}{u_0^2 + u_1^2} - \frac{u_1(\dot{u}_2u_0 - \dot{u}_0u_2)}{u_0^2 - u_2^2}\right] + (\dot{u}\cdot\partial_u)f + c\cdot u - m\|u\|,$$

where the arbitrary function $f(u)$ satisfies the condition $u\cdot\partial_u f = 0$. In general, suppose that the pseudo-Euclidean space has a dimension greater than two and a signature not equal to two. An invariant Lagrangian, for which the Euler–Poisson equations are third-order equations, will then be missing in this space.

Let X be the generator of the Lorentz transformations which is defined by the vector parameters ω and β, and let the expression $\mathcal{E}$ correspond to Eq. (11). Then $X_3(\mathbf{E}) = \omega\times\mathbf{E} + (\beta\cdot\mathbf{v})\mathbf{E} - (\mathbf{v}\cdot\mathbf{E})\beta$. Let X be the generator of the pseudo-orthogonal transformations in three-dimensional space, which is defined by the vector parameter w, X_3^T, its continuation into the space $T_3\mathbf{R}^3$, and let the expression $\mathcal{E}$ correspond to Eq. (10). Then $X_3^T(\mathcal{E}) = w\times\mathcal{E}$. We thus conclude that the pseudo-orthogonal transformations and the Lorentz transformations are not the generalized invariance transformations.[12] The proposed method for finding the invariant Euler–Poisson equations is thus much more general than the methods arising from the Lagrangian.

We showed that the Mathiesson equation and the equation for the geodesic circles in some cases can be considered in the context of Ostrogradskii mechanics and Kawaguchi geometry. The case of the two-dimensional space was considered in Ref. 14.

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Investigation of the solution of a nonlinear equation of the gravitation theory

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In the present paper, we study the solution of a nonlinear equation of gravitation theory.[1] This problem has been formulated by Academician A. A. Logunov in connection with the new approach he developed to describe the gravitational interaction in Minkowski space.[2]

One version of the gravitational field theory which uses this approach is based on an incomplete geometrized density of the Lagrangian

$$L = -\frac{c^4\sqrt{-g}}{16\pi G} g^{ik}[\Pi^m_{il}\Pi^l_{km} - \Pi^l_{ik}\Pi^m_{lm}] + L_M(g_{ik}, \varphi_A),$$

where the relation $g_{ik}=\gamma_{ik}+\varphi_{ik}$ is the Riemann space–time metric tensor g_{ik} with the metric tensor γ_{ik} of the Minkowski space and with the gravitational field φ_{ik}, G is the gravitational constant, c is the speed of light, $L_M(g_{ik}, \varphi_A)$ is the density of the Lagrangian for matter, which depends on the metric tensor g_{ik} and on the remaining matter fields φ_A, and Π^m_{il} is the tensor which is obtained from the Christoffel symbols through replacement of the partial derivatives by the covariant derivatives with respect to the metric γ_{ik}. Furthermore, new physical conditions, which allow one to limit the spin states of the gravitational field, have been introduced in this theory.

In the approach developed by Logunov, the laws of the conservation of energy and momentum and of angular momentum are valid for a closed system. Ten gravitational field equations have been obtained in this theory; the first six are the Hilbert–Einstein equations, and the four others reflect the nature of the gravitational field. In analyzing the different solutions of this theory, the problem arises of determining the functional dependence of the coordinates of Riemann space–time on the coordinates of Minkowski space. Knowing this dependence, one can obtain some very important physical data about the distribution of the gravitational field in space, about the nature of the forces acting on a test particle, the time for it to fall onto the force center, etc.

From the mathematical point of view, the problem of determining the functional dependence of the Riemann space–time coordinates on the coordinates of Minkowski space reduces to the solution of a system of non-linear differential equations. In particular, in the case of a static, spherically symmetric gravitational field, this system is reduced to a single equation

$$y'' + \frac{2y'}{x} - \frac{(y')^2 m}{y(y-2m)} - \frac{(y-2m)}{y}\left[\frac{m}{y^2} + \frac{2y}{x^2}\right] = 0, \qquad (1)$$

where m is the mass of the spherically symmetric body, and x is its radius.

The problem consists of finding a solution of this equation which satisfies the condition $y(x)/x \to 1$ for $x \to \infty$, which is necessary to satisfy Newton's law of gravitation (the external condition). Equation (1) was analyzed in Ref. 1 for the boundary conditions $(m=2)$

$$y(0) = 2; \quad y(x) > 2 \quad \text{for all} \quad x \in (0, \infty); \qquad \lim_{x\to\infty} \frac{y(x)}{x} = 1, \qquad (2)$$

where $y(x) \in C^0[0, \infty) \cap C^2(0, \infty)$. The following theorem for the existence of a global solution of Eqs. (1) and (2) was proved in Ref. 1.

Theorem. The solution y(x) of Eqs. (1) and (2) exists along the entire semiaxis $[0, \infty)$, where y(x) is an increasing function on the half-line $[0, \infty)$.

We emphasize that a numerical calculation of the solution y(x) was carried out on a computer in Ref. 1. This calculation showed that at large x the solution y(x) behaves as x+1, i.e.,

$$y(x) \sim x + 1.$$

We are thus led to ask whether the global solution of the boundary value problem in (1) and (2) is unique and whether there is a more accurate asymptotic behavior of the solution y(x).

In this paper we prove the following two theorems.

Theorem 1. The solution $y(x) \in C^2(a, \infty)$ of Eq. (1) with m = 1 on the semiaxis (a, ∞),

