DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 547–551

HAMILTON—OSTROHRADS'KYJ APPROACH TO RELATIVISTIC FREE SPHERICAL TOP DYNAMICS

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ABSTRACT. Dynamics of classical spinning particle in special relativity with Pirani constraint is a typical example of the generalized Hamilton theory recently developed by O. Krupková and discovers some characteristic features of the latter.

Recent developments in Ostrohrads'kyj mechanics, in particular, some substantial progress in understanding its Hamiltonian counterpart, may give rise to the enrichment in the family of the generalized canonical dynamical systems which describe certain processes in the real physical world. In this report we call upon the Reader to follow the possibility of building yet another canonical model of the free spinning particle motion in special relativity. One way to do that is to start with the system of Dixon's equations [1] in flat space-time

$$\begin{cases} \dot{\mathfrak{P}} = \mathfrak{o} & \text{(1a)} \\ \dot{\mathfrak{S}} = 2\,\mathfrak{P} \wedge \mathfrak{u}\,. & \text{(1b)} \end{cases}$$

The four-vector $\mathfrak{P}$, the velocity four-vector $\mathfrak{u}$, and the symmetric tensor $\mathfrak{S}$ do not constitute a complete system of variables if one wishes to put the equations (1) into the Hamiltonian form in the usual way. On the other hand, the system (1) is under-determined and needs to be supplemented by some constraints.

A profound classical and quantum description of the relativistic top dynamics based upon the Dirac theory of constraints was offered by A. J. Hanson and T. Regge in [2], where they exploited the constraint $\mathfrak{P}_{\mathfrak{q}}\mathfrak{S}^{\mathfrak{p}\mathfrak{q}} = \mathfrak{o}$, sometimes referred to as the Tulczyjew supplementary condition. At the same time, some relativistic centre-of-mass considerations, concerning the dipole model of massive spinning particle in relativity (see [3] and references therein), bring about the alternative supplementary condition,

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1991 *Mathematics Subject Classification*. 70H05, 83A05, 49N45.

Key words and phrases. Higher-order mechanics, Generalized Hamilton equations, Inverse variational problem, Invariance, Relativistic top, Classical spin.

This paper is in final form and no version of it will be submitted for publication elsewhere .

$$\mathfrak{u}_q \mathfrak{S}^{pq} = \mathfrak{o}\,, \tag{2}$$

sometimes named Pirani supplementary condition. In the present report I shall make an attempt to 'hamiltonize' the ideology of the Pirani constraint in contrast to what was already done with respect to the Tulczyjew one.

A brief overview of classical spinning particle settlement

In the presence of the supplementary condition (2) it is possible to re-solve with respect to $\mathfrak{S}$ the following definition of the spin four-vector $\mathfrak{s}$,

$$\mathfrak{s}_p = \frac{1}{2\|\mathfrak{u}\|} \varepsilon_{mnqp} \mathfrak{u}_m \mathfrak{S}^{nq}\,,$$

and in this way the system of equations (1b,2) may be replaced by the following one:

$$\begin{cases} \mathfrak{P} = \mu_0 \dfrac{\mathfrak{u}}{\|\mathfrak{u}\|} + \dfrac{*\,\dot{\mathfrak{u}} \wedge \mathfrak{u} \wedge \mathfrak{s}}{\|\mathfrak{u}\|^3} & \text{(3a)} \\ \dot{\mathfrak{s}} \wedge \mathfrak{u} = \mathfrak{o} & \text{(3b)} \\ \mathfrak{s} \cdot \mathfrak{u} = \mathfrak{o}\,. & \text{(3c)} \end{cases}$$

The quantity $\mu_0 = \frac{\mathfrak{P}\cdot\mathfrak{u}}{\|\mathfrak{u}\|}$ entering in the expression (3a) may immediately be shown to constitute an integral of motion (even if we replaced the right-hand side of (1a) by some force $\mathfrak{F}$, provided only the condition $\mathfrak{F} \cdot \mathfrak{u} = \mathfrak{o}$ is obeyed). The equation (3b) may also be given an equivalent form of

$$\dot{\mathfrak{s}} = \frac{\dot{\mathfrak{s}} \cdot \mathfrak{u}}{\|\mathfrak{u}\|^2}\, \mathfrak{u}\,, \tag{4}$$

by means of which we deduce from (3a) that the value of the contraction $\dot{\mathfrak{P}} \cdot \mathfrak{s} \equiv (\mathfrak{P} \cdot \mathfrak{s})^{\cdot} - \mathfrak{P} \cdot \dot{\mathfrak{s}}$ in fact equals $\frac{\mu_0}{\|\mathfrak{u}\|}\, \dot{\mathfrak{s}} \cdot \mathfrak{u}$, and thus by means of (4) again, the spin four-vector $\mathfrak{s}$ is constant everywhere where $\dot{\mathfrak{P}}\cdot\mathfrak{s}$ is null; hence in the flat space-time there is no precession due to (1a), i.e.

$$\dot{\mathfrak{s}} = \mathfrak{o}\,. \tag{5}$$

The third-order equation of motion, obtained by substituting (3a) into (1a), coincides, within the realm of the Pirani supplementary condition, with the equation, suggested by Mathisson [4] in terms of $\mathfrak{S}$.

Now let us fix the parametrization of the world line of the particle by means of choosing the coordinate time as the parameter along the trajectory. We introduce the space vs. time splitting of the variables with the help of the following notations:

$$\mathfrak{u} = (1, \mathbf{v}); \quad \mathfrak{P} = (\mathfrak{P}_0, \mathbf{P}); \quad \mathfrak{s} = (\mathfrak{s}_0, \mathbf{s}),$$

by which the formulae (3a) and (3c) take the shape (please notice $\mathbf{v}^2 = \mathbf{v}\cdot\mathbf{v} = \mathsf{v}_a \mathsf{v}^a = -\sum_{a=1}^{3} \mathsf{v}_a \mathsf{v}_a$, although all constructions have the same appearance independent of the signature of the (pseudo-) Euclidean metrics)

$$\begin{cases} \mathbf{P} = \dfrac{\mu_0}{\sqrt{1+\mathbf{v}^2}}\,\mathbf{v} + \dfrac{1}{(1+\mathbf{v}^2)^{3/2}}\left(\mathbf{v}' \times \mathbf{s} - \mathfrak{s}_0 \cdot \mathbf{v}' \times \mathbf{v}\right) & (6) \\ \mathfrak{s}_0 + \mathbf{s}\cdot\mathbf{v} = 0\,. & (7) \end{cases}$$

Hamiltonian dynamics of free relativistic top

We shall follow the approach of [5] and describe the Hamiltonian dynamics by means of the kernel of the Lepagean differential two-form

$$-dH \wedge dt + d\mathsf{p}_a \wedge d\mathsf{x}^a + d\mathsf{p}'_a \wedge d\mathsf{v}^a \tag{8}$$

with the Hamilton function

$$\cancel{H = \mathbf{p}\cdot\mathbf{v}}\,. \tag{9}$$

One observation consists in that it is possible to define such functions $\mathbf{p}$ and $\mathbf{p}'$ of the variables $\mathbf{v}$ and $\mathbf{v}'$, that in (8) all one-contact terms of the second order cancel out, and the expression (8) becomes

$$-\frac{\partial \mathsf{p}_a}{\partial \mathsf{v}^b}\,\omega^a \wedge \omega'^b - \frac{\partial \mathsf{p}'_a}{\partial \mathsf{v}^b}\,\omega'^a \wedge \omega'^b - \frac{\partial \mathsf{p}_a}{\partial \mathsf{v}^b}\mathsf{v}'^b\,\omega^a \wedge dt \cancel{+} \frac{\partial \mathsf{p}_a}{\partial \mathsf{v}'^b}\,\omega^a \wedge d\mathsf{v}'^b\,,$$

with $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ denoting the contact forms of the first and of the second order resp. The functions $\mathbf{p}$ and $\mathbf{p}'$ constitute the generalized Legendre transformation, and the Lagrangian counterpart of dynamics is described by the Euler-Poisson expression

$$-\frac{d}{dt}\,\mathbf{p} \doteqdot -\left(\mathbf{v}'.\,\partial_{\mathbf{v}} + \mathbf{v}''.\,\partial_{\mathbf{v}'}\right)\mathbf{p}\,. \tag{10}$$

We can suggest the following expression of the Legendre transformation which I believe points at the adequate way of the hamiltonization of the dynamics, governed by the system of equations (1a & 3a),

$$\begin{cases} \mathbf{p} = \dfrac{M_0}{(\mathfrak{s}_0{}^2 + \mathbf{s}^2)^{3/2}}\,\dfrac{\mathbf{v}}{\sqrt{1+\mathbf{v}^2}} + \dfrac{\mathbf{v}' \times (\mathbf{s} - \mathfrak{s}_0 \mathbf{v})}{\left[(\mathbf{s} - \mathfrak{s}_0\mathbf{v})^2 + (\mathbf{s}\times\mathbf{v})^2\right]^{3/2}} & \text{(11a)} \\ \mathbf{p}' = \dfrac{\boldsymbol{\xi} \times (\mathbf{s} - \mathfrak{s}_0\mathbf{v})}{3\left(\mathfrak{s}_0{}^2 + \mathbf{s}^2\right)\left[(\mathbf{s} - \mathfrak{s}_0\mathbf{v})^2 + (\mathbf{s}\times\mathbf{v})^2\right]^{1/2}}\,, & \text{(11b)} \end{cases}$$

where

$$\xi_a = \frac{1}{\mathfrak{s}_0} \frac{(\mathfrak{s}_0 + \mathbf{s} \cdot \mathbf{v})\, \mathrm{s}_a - ({\mathfrak{s}_0}^2 + \mathbf{s}^2)\, \mathrm{v}_a}{(\mathbf{s} - \mathfrak{s}_0 \mathbf{v})^2 - (\mathrm{s}_a - \mathfrak{s}_0 \mathrm{v}_a)^2 + (\mathbf{s} \times \mathbf{v})^2}\,, \tag{12}$$

and M_0 is some constant number.

Looking closer at the expression (10) with $\mathbf{p}$ given by (11a), convinces that the Lagrange system, defined by (10), carries along the primary semispray constraint (if stick to the terminology of [5])

$$\frac{M_0}{\mathfrak{s}_0 - \mathbf{s}^2} \left[\frac{\mathbf{s} \cdot \mathbf{v}'}{\sqrt{1 + \mathbf{v}^2}} - \frac{(\mathfrak{s}_0 + \mathbf{s} \cdot \mathbf{v})(\mathbf{v} \cdot \mathbf{v}')}{(1 + \mathbf{v}^2)^{3/2}} \right] = 0\,. \tag{13}$$

The expression within square brackets in (13) presents an exact total derivative, so we obtain the first integral of motion,

$$\frac{\mathfrak{s}_0 + \mathbf{s} \cdot \mathbf{v}}{\sqrt{1 + \mathbf{v}^2}}\,, \tag{14}$$

that clearly generalizes the genuine constraint (7) which in turn—we recall—is nothing else but the rudiment of the Pirani supplementary condition (2).

One would have to prove that the Hamiltonian dynamics defined by (8,9,11) really has some connection with the classical spinning particle dynamics given by (1a,3a,3c, and 5). This connection clears up in two steps. First, prove the following algebraic identity:

$$\frac{(\mathbf{s} - \mathfrak{s}_0 \mathbf{v})^2 + (\mathbf{s} \times \mathbf{v})^2}{({\mathfrak{s}_0}^2 + \mathbf{s}^2)(1 + \mathbf{v}^2)} \equiv 1 - \frac{(\mathfrak{s}_0 + \mathbf{s} \cdot \mathbf{v})^2}{({\mathfrak{s}_0}^2 + \mathbf{s}^2)(1 + \mathbf{v}^2)}\,. \tag{15}$$

Then, multiply (11a) by the constant of motion $\left[\dfrac{(\mathbf{s} - \mathfrak{s}_0 \mathbf{v})^2 + (\mathbf{s} \times \mathbf{v})^2}{1 + \mathbf{v}^2} \right]^{3/2}$ and compare with (6) to conclude that there must exist a link-up between the constants μ_0 and M_0:

$$\mu_0 = M_0 \left[1 - \frac{(\mathfrak{s}_0 + \mathbf{s} \cdot \mathbf{v})^2}{({\mathfrak{s}_0}^2 + \mathbf{s}^2)(1 + \mathbf{v}^2)} \right]^{3/2} \tag{16}$$

We may summarize the results of the preceding calculations in a couple of statements:

(1) As far as Pirani supplementary condition is recognized, the phase space of the free classical spinning particle may be augmented in the way that the dynamics allows a generalized Hamiltonian description with the Hamilton function

$$H = \frac{M_0}{({\mathfrak{s}_0}^2 + \mathbf{s}^2)^{3/2}} \frac{\cancel{\mathbf{v}}^2}{\sqrt{1 + \mathbf{v}^2}} - \frac{[\mathbf{v}', \mathbf{v}, \cancel{\mathbf{v}}]}{\left[(\mathbf{s} - \mathfrak{s}_0 \mathbf{v})^2 + (\mathbf{s} \times \mathbf{v})^2\right]^{3/2}}\,; \tag{17}$$

(2) The mass μ_0 of the 'hamiltonized' particle depends upon its spin according to the expression (16); it is worthwhile to mention at this place that the Hamiltonian description of [2] demanded an arbitrary dependence of the particle's mass on its spin;

(3) Any dynamical subsystem, obtained by prescribing a fixed value to the integral of motion (14), never is Hamiltonian by itself; in particular, we could not have obtained a variational description of the spinning particle motion if the constant of motion (14) had been frozen by means of the equation (7) or, equivalently, by the demand that μ_0 and M_0 take the same value in (16);

(4) The Legendre transformation, given by (11), is not globally defined in an intrinsic sense, as may be seen from (12); nevertheless, the Hamilton function is defined quite nicely via the expression (17).

Guessing the form of the Legendre transformation (11) is equivalent to solving the Poincaré-invariant inverse problem of calculus of variations in order 3. That was treated in [6] and the corresponding Euler-Poisson expression (10) found. But I did not know the appropriate expression for the Legendre transformation until 1995 when a set of Lagrange functions corresponding to (10) was discovered [7].

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