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ON THE GENERALIZED WEAK HARNACK INEQUALITY FOR NON-NEGATIVE SUPER-SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS WITH ABSORPTION TERM

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We prove the generalized weak Harnack inequality for non-negative super-solutions to quasilinear elliptic equations with the generalized Orlicz growth

$$\operatorname{div}\Big(\Phi(x,|\nabla u|)\frac{\nabla u}{|\nabla u|^2}\Big) = b(x)f(u), \;\; b(x)\geqslant 0, \quad x\in\Omega, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n\geqslant 2$.

Throughout the paper we suppose that $\Phi(x,v) : \Omega\times\mathbb{R}_+ \to \mathbb{R}_+$ is a non-negative function satisfying the following properties: for any $x\in\Omega$ the function $v \to \dfrac{\Phi(x,v)}{v}$ is non-decreasing and $\lim\limits_{v\to 0}\Phi(x,v) = 0$, $\lim\limits_{v\to+\infty}\Phi(x,v)=+\infty$. We also assume that

$$a(x)G(v) \leqslant \Phi(x,v) \leqslant b(x)G(v), \quad G(v) := \int\limits_0^v g(s)\,ds, \tag{2}$$

where $a(\cdot)\geqslant 0$, $b(\cdot)\geqslant 0$ and $g(\cdot)>0$.

And moreover, we also assume that $u\to f(u)$ is increasing and (F_δ)there exists $\delta>0$ such that

$$(1+\delta)F(u)\leqslant f(u)\;u, \quad F(u) := \int\limits_0^u f(s)\,ds, \quad u>0.$$

The function $f(u) = u^\delta\, f_1(u)$, $\delta>0$ satisfies condition (F_δ), provided that $f_1(u)$ is non decreasing. We note that in the case $\delta=0$, condition (F_δ) is a consequence of the fact that f is increasing.

The aim of this paper is to prove the weak Harnack inequality for non-negative super-solutions to equation (1). Before formulating our main result, let us remind the reader of the definition of a weak super-solution to equation

(1). We write $W^{1,\Phi}(\Omega)$ for the class of functions $u \in W^{1,1}(\Omega)$ with $\int\limits_{\Omega} \Phi(x, |\nabla u|)dx < \infty$ and we say that u is a weak super-solution to equation (1) if $u \in W^{1,\Phi}_{loc}(\Omega)$ and the following integral inequality

$$-\int\limits_{E} \Phi(x, |\nabla u|)\frac{\nabla u}{|\nabla u|^2}\nabla\varphi\, dx \leqslant \int\limits_{E} b(x)f(u)\varphi\, dx \qquad (3)$$

holds for every open set $E \subset \Omega$ and for all non-negative $\varphi \in W^{1,\Phi}_0(E)$.

To formulate our results set

$$\mu(x_0, r) := \Big(\frac{1}{B_r(x_0)}\int\limits_{B_r(x_0)} [a(x)]^{-t}dx\Big)^{\frac{1}{t}}\Big(\frac{1}{B_r(x_0)}\int\limits_{B_r(x_0)} [b(x)]^{m}dx\Big)^{\frac{1}{m}}$$

In what follows we assume that the function $r \to \mu(x_0, r)$ is non increasing. Our main result of this paper reads as follows.

Теорема 1. *Let u be a weak non-negative super-solution to equation* (1) *and let conditions* (2) *and* (F_δ) *be fulfilled. Denote $m_\rho := \inf\limits_{B_\rho(x_0)} u$, then there exist constants C, $C_1 > 1$, depending only on n, p, q, t, m, δ such that for any $\theta \in \big(0, \frac{\min(p-1,\delta)nt}{n(t+1)-pt}\big)$ there holds*

$$\frac{1}{B_r(x_0)}\int\limits_{B_\rho(x_0)} exp\Big(\theta\int\limits_{2m_\rho}^{2u}\frac{dt}{t+\rho G^{-1}\big(F(t)\big)}\Big)dx \leqslant exp\big(C[\mu(x_0,\rho)]^{C_1}\big), \qquad (4)$$

provided that $B_{8\rho}(x_0) \subset \Omega$. Here $G^{-1}(\cdot)$ is the inverse $G(\cdot)$.

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ПРО УЗАГАЛЬНЕНУ СЛАБКУ НЕРІВНІСТЬ ГАРНАКА ДЛЯ НЕВІД'ЄМНИХ СУПЕРРОЗВ'ЯЗКІВ КВАЗІЛІНІЙНИХ ЕЛІПТИЧНИХ РІВНЯНЬ ІЗ ЧЛЕНОМ ПОГЛИНАННЯ

Доведено узагальнену слабку нерівність Гарнака для невід'ємних суперрозв'язків квазілінійних еліптичних рівнянь із узагальненим зростанням Орліча

$$div\Big(\Phi(x, |\nabla u|)\frac{\nabla u}{|\nabla u|^2}\Big) = b(x)f(u), \ \ b(x) \geqslant 0.$$