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WELL-POSEDNESS OF LORD – SHULMAN THERMOPIEZOELECTRICITY VARIATIONAL PROBLEM

On the basis of initial boundary value Lord – Shulman thermopiezoelectricity problem we formulate the corresponding variational problem in terms of vector of elastic displacements, electric potential, temperature increment and vector of heat fluxes. Using energy balance equation of the variational problem, we establish the sufficient conditions for regularity of input data of the problem and prove the uniqueness of its solution. For proving the existence of the general solution of the problem we use Galerkin semidiscretization by spatial variables and show that the limit of the sequence of its approximations is a solution of variational problem of Lord – Shulman thermopiezoelectricity. This fact allows us to construct a reasonable procedure for calculation an approximation of the solution of this problem.

The classical theory of linear thermopiezoelectricity describes the interaction of thermal, electrical and mechanical fields in pyroelectric material and was introduced by R. D. Mindlin [17]. The further study of this theory was performed by W. Nowacki [18]. Eliminating the electrical field from the scope, we obtain a classical thermoelasticity model. The main drawback of the latter theory (and therefore the classical thermopiezoelectricity theory too) is the assumption of infinite speed of propagation of thermal signals in the materials. To overcome it, Lord and Shulman [16] proposed a modified theory of thermoelasticity (LS-theory), where the classical Fourier's law of heat conduction is replaced by Maxwell – Cattaneo equation with introduction of so-called «relaxation time». Similar generalizations of the thermoelasticity model can be found in [2]. Chandrasekharaiah was the first researcher to apply the LS-theory to thermopiezoelectricity [10]. Nowadays a some few generalized theories of thermoelasticity and thermopiezoelectricity is known, namely Green – Lindsay, Chandrasekharaiah – Tzou, Green – Naghdi etc. A comprehensive review of the existing generalization theories can be found in [8, 11, 13, 14]. Researchers used different techniques to the solutions of the generalized thermopiezoelectricity problems (see [9, 12, 19]).

In authors' previous works [3, 5, 6, 20] the classical thermopiezoelectricity problem was considered. In our paper [21] forced vibrations of pyroelectric materials under LS-theory has been studied. In this article, using similar techniques as in [7], we construct the corresponding variational problem for LS-theory of thermopiezoelectricity and prove its well-posedness.

In Section **1** the initial boundary value problem of Lord – Shulman thermopiezoelectricity is described. Section **2** is dedicated to construction of the corresponding variational problem. Energy balance law is obtained in Section **3** and it is then used as an important technique for investigation of the variational problem. In Section **4** a priori energy estimates are obtained by means of transformations applied to energy balance law. In order to prove the existence of variational problem solution, finite element semidiscretization is done as described in Section **5**. Finally, in Section **6** the well-posedness of the variational problem is proved.

1. Initial boundary value problem of Lord – Shulman thermopiezoelectricity. Let Ω be the bounded connected domain of points $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ with Lipschitz continuous boundary $\partial\Omega = \Gamma$, and $\mathbf{n} = \{n_i\}_{i=1}^d$ is a unit outer normal vector, $n_i = \cos(\widehat{\mathbf{n}, x_i})$. Also let us consider time interval $[0, T]$, $0 < T < +\infty$. Our goal is to find vector of elastic displacements $\mathbf{u} = \{u_i(\mathbf{x}, t)\}_{i=1}^d$, electric potential $p = p(\mathbf{x}, t)$, temperature change $\theta = \theta(\mathbf{x}, t)$, and heat flux vector $\mathbf{q} = \{q_i(\mathbf{x}, t)\}_{i=1}^{d}$, which satisfy the following equations in $\Omega \times (0, T]$ (here and everywhere below the ordinary summation over repetitive indices is assumed):

$$\rho(u_i'' - f_i) - \sigma_{ij,j} = 0 , \tag{1}$$

$$D'_{k,k} + J_{k,k} = 0 , \tag{2}$$

$$\rho(T_0 S' - w) + q_{i,i} = 0 , \tag{3}$$

$$t_0 q_i' + q_i = -\lambda_{ij}\theta_{,j} . \tag{4}$$

The above expressions (1)–(4) are equation of motion, Maxwell's equation in differential form, heat conduction equation and modified Fourier's law (also known as Maxwell – Cattaneo equation). The parameter $t_0 > 0$ is a so-called «relaxation time». Putting $t_0 = 0$ into equation (4) eliminates heat flux q from the set of independent variables and we come to the equations of the classical thermopiezoelectricity problem [5, 6].

In the equations (1)–(4) are used

– the constitutive relations for stress tensor

$$\sigma_{ij} = c_{ijkm}\varepsilon_{km}(\mathbf{u}) - c_{ijkm}\alpha_{km}\theta + a_{ijkm}\varepsilon_{km}(\mathbf{u}') - e_{kij}E_k(p) , \tag{5}$$

– electric displacement vector

$$D_k = \chi_{km}E_m(p) + e_{kij}\varepsilon_{ij}(u) + \pi_k\theta , \tag{6}$$

– entropy density

$$\rho S = \rho c_v T_0^{-1}\theta + c_{ijkm}\alpha_{km}\varepsilon_{ij}(\mathbf{u}) + \pi_k E_k(p) . \tag{7}$$

Vector J_k is the electric current density, generated by a free electric charge density. We assume that pyroelectric material is not an ideal dielectric, and the electric current flows through the pyroelectric specimen and satisfies standard Ohm's law, i.e.

$$J_k = z_{km}E_m(p) . \tag{8}$$

Strain tensor $\boldsymbol{\varepsilon}_{km}$ and electric field vector E_k are assumed to satisfy the relations

$$\boldsymbol{\varepsilon}_{km}(\mathbf{u}) = \frac{1}{2}(u_{k,m} + u_{m,k}) ,$$

$$E_k(p) = -p_{,k} , \tag{9}$$

where comma in the subscript stands for the partial derivative by the spatial variable, i.e. $g_{,k} = \partial g/\partial x_k$.

In (5)–(8) notation ρ is a mass density of pyroelectric material, c_v is its specific heat and T_0 is a fixed uniform reference temperature of the specimen. Notation f_i is a vector of volume mechanical forces and w represents volume heat forces, a_{ijkm} and c_{ijkm} are viscosity and elasticity coefficients of a pyroelectric material with the common properties of symmetry and ellipticity, and e_{kij} are coefficients of piezoelectricity tensor with property

$$e_{kij} = e_{kji} .$$

Coefficients z_{km}, χ_{ij}, λ_{ij}, α_{km} are the symmetrical and elliptical electric conductivity, dielectric susceptibility, heat conductivity and thermal expansion coefficients respectively. π_k are pyroelectric coefficients, which satisfy the inequality [1, 18]

$$\chi_{km} y_k y_m + 2\pi_k y_k \xi + \rho c_v \xi^2 \ge 0 \qquad \forall \xi, y_k \in \mathbb{R}.$$

The system of partial differential equations (1)–(4) is complemented by boundary conditions

$$
\begin{aligned}
& u_i = 0 && \text{on} && \Gamma_u \times [0,T], \quad \Gamma_u \subset \Gamma, \quad \operatorname{mes}(\Gamma_u) > 0, \\
& \sigma_{ij} n_j = \bar{\sigma}_i && \text{on} && \Gamma_\sigma \times [0,T], \quad \Gamma_\sigma = \Gamma \setminus \Gamma_u, \\
& p = 0 && \text{on} && \Gamma_p \times [0,T], \quad \Gamma_p \subset \Gamma, \quad \operatorname{mes}(\Gamma_p) > 0, \\
& (D'_k + J_k) n_k = 0 && \text{on} && \Gamma_d \times [0,T], \quad \Gamma_d \subset \Gamma, \quad \Gamma_d \cap \Gamma_p = \varnothing, \\
& \int_{\Gamma_e} (D'_k + J_k) n_k \, d\gamma = I && \text{on} && \Gamma_e \times [0,T], \quad \Gamma_e = \Gamma \setminus (\Gamma_d \cap \Gamma_p), \\
& E_k(p) - n_k E_m(p) n_m = 0 && \text{on} && \Gamma_e \times [0,T], \\
& \theta = 0 && \text{on} && \Gamma_\theta \times [0,T], \quad \Gamma_\theta \subset \Gamma, \quad \operatorname{mes}(\Gamma_\theta) > 0, \\
& q_i n_i = 0 && \text{on} && \Gamma_q \times [0,T], \quad \Gamma_q = \Gamma \setminus \Gamma_\theta,
\end{aligned}
\tag{10}
$$

and the initial conditions

$$
\begin{aligned}
& \mathbf{u}\big|_{t=0} = \mathbf{u}_0, \qquad \mathbf{u}'\big|_{t=0} = \mathbf{v}_0, \qquad p\big|_{t=0} = p_0, \\
& \theta\big|_{t=0} = \theta_0, \qquad \mathbf{q}\big|_{t=0} = \mathbf{q}_0 \ \text{ in } \Omega.
\end{aligned}
\tag{11}
$$

Here $\bar{\boldsymbol{\sigma}} = \{\bar{\sigma}_i(\mathbf{x},t)\}_{i=0}^{d}$ and $I = I(\mathbf{x},t)$ represent vector of mechanical loading and external electric current, respectively.

Also, for convenience, similar to [10], we introduce artificial coefficients $æ_{ij}$ to satisfy the following condition:

$$T_0 æ_{ij} \lambda_{jm} = \delta_{im},$$

where δ_{im} are the elements of the unit matrix. Coefficients $æ_{ij}$ also satisfy ellipticity conditions. Then the modified Fourier's law (4) can be rewritten in the following form:

$$t_0 æ_{ij} q'_i + æ_{ij} q_i = -T_0^{-1} \theta_{,j}. \tag{12}$$

We will use equation (12) instead of (4) in our further analysis.

2. Variational problem statement. Let us introduce the spaces of admissible elastic displacements, electric potentials, temperature increments (relative to the initial temperature T_0), and heat fluxes, respectively:

$$
\begin{aligned}
& \mathbf{V} = \{\mathbf{v} \in [H^1(\Omega)]^d : \mathbf{v} = 0 \quad \text{on} \quad \Gamma_u\}, \\
& P = \{r \in H^1(\Omega) : r = 0 \quad \text{on} \quad \Gamma_p, \quad r = \text{const} \quad \text{on} \quad \Gamma_e\}, \\
& Z = \{\zeta \in H^1(\Omega) : \zeta = 0 \qquad \text{on} \quad \Gamma_\theta\}, \\
& \mathbf{H} = \{y \in H(\operatorname{div};\Omega) : y_i, \operatorname{div} \mathbf{y} \in L^2(\Omega), y_i n_i = 0 \quad \text{on} \quad \Gamma_q\}.
\end{aligned}
$$

We denote $\boldsymbol{\Phi} := \mathbf{V} \times P \times Z \times \mathbf{H}$ and the dual space $\boldsymbol{\Phi}' := \mathbf{V}' \times P' \times Z' \times \mathbf{H}'$.

The initial boundary value problem of thermopiezoelectricity (1)–(3), (5)–(12) admits the following variational formulation:

$$
\begin{aligned}
&\textit{find:}\qquad \boldsymbol{\psi} = (\mathbf{u}, p, \theta, \mathbf{q}) \in L^2(0,T;\boldsymbol{\Phi}) \qquad \textit{such that}\\
&m(\mathbf{u}''(t),\mathbf{v}) + a(\mathbf{u}'(t),\mathbf{v}) + c(\mathbf{u}(t),\mathbf{v}) - e(p(t),\mathbf{v}) -\\
&\qquad\qquad - \beta(\theta(t),\mathbf{v}) = \langle \mathbf{l}_\sigma(t),\mathbf{v}\rangle,\\
&\chi(p'(t),r) + z(p(t),r) + \pi(\theta'(t),r) + e(r,\mathbf{u}'(t)) = \langle \ell_e(t), r\rangle,\\
&s(\theta'(t),\zeta) + T_0^{-1}(\operatorname{div}\mathbf{q}(t),\zeta) +\\
&\qquad\qquad + \pi(\zeta,p'(t)) + \beta(\zeta,\mathbf{u}'(t)) = \langle \ell_\vartheta(t),\zeta\rangle,\\
&t_0 æ(\mathbf{q}'(t),\mathbf{y}) - T_0^{-1}(\operatorname{div}\mathbf{y},\theta(t)) + æ(\mathbf{q}(t),\mathbf{y}) = 0 \qquad \forall t\in(0,T],\\
&m(\mathbf{u}'(0)-\mathbf{v}_0,\mathbf{v}) = 0, \qquad c(\mathbf{u}(0)-\mathbf{u}_0,\mathbf{v}) = 0,\\
&\chi(p(0)-p_0,r) = 0, \qquad s(\theta(0)-\theta_0,\zeta) = 0,\\
&æ(\mathbf{q}(0)-\mathbf{q}_0,\mathbf{y}) = 0 \qquad \forall\boldsymbol{\psi} = (\mathbf{v},r,\zeta,\mathbf{y})\in\boldsymbol{\Phi},
\end{aligned}
\tag{13}
$$

where bilinear and linear forms are defined by the following expressions:

$$
\begin{aligned}
&m(\mathbf{u},\mathbf{v}) = \int_\Omega \rho u_i v_i\,dx, \qquad a(\mathbf{u},\mathbf{v}) = \int_\Omega a_{ijkm}\varepsilon_{ij}(\mathbf{u})\varepsilon_{km}(\mathbf{v})\,dx,\\
&c(\mathbf{u},\mathbf{v}) = \int_\Omega c_{ijkm}\varepsilon_{ij}(\mathbf{u})\varepsilon_{km}(\mathbf{v})\,dx,\\
&e(r,\mathbf{v}) = \int_\Omega e_{kij}E_k(r)\varepsilon_{ij}(\mathbf{v})\,dx \qquad \forall\,\mathbf{u},\mathbf{v}\in\mathbf{V},\\
&z(p,r) = \int_\Omega z_{km}E_k(p)E_m(r)\,dx,\\
&\chi(p,r) = \int_\Omega \chi_{km}E_k(p)E_m(r)\,dx \qquad \forall p,r\in P,\\
&\beta(\zeta,\mathbf{v}) = \int_\Omega c_{ijkm}\alpha_{km}\varepsilon_{ij}(\mathbf{v})\zeta\,dx,\\
&\pi(\zeta,r) = \int_\Omega \pi_k E_k(r)\zeta\,dx,\\
&s(\theta,\zeta) = \int_\Omega \rho c_v T_0^{-1}\theta\zeta\,dx \qquad \forall\zeta\in Z,\\
&æ(\mathbf{q},\mathbf{y}) = \int_\Omega æ_{\ell m} q_\ell y_m\,dx \qquad \forall\mathbf{q},\mathbf{y}\in\mathbf{H},\\
&\langle \mathbf{l}_\sigma,\mathbf{v}\rangle = \int_\Omega \rho f_i v_i\,dx + \int_{\Gamma_\sigma}\sigma_i v_i\,d\gamma \qquad \forall\,\mathbf{v}\in\mathbf{V},\\
&\langle \ell_e, r\rangle = -Ir\big|_{\Gamma_e} \qquad \forall\, r\in P, \qquad \langle \ell_\vartheta,\zeta\rangle = \int_\Omega T_0^{-1}\rho w\zeta\,dx \qquad \forall\zeta\in Z.
\end{aligned}
\tag{14}
$$

Here $\operatorname{div}\mathbf{y} := y_{i,i}$ for each vector valued function $\mathbf{y} = \mathbf{y}(\mathbf{x}) \in H^1(\Omega)$, and

$$
(z,w) = \int_\Omega zw\,dx \qquad \forall z,w\in L^2(\Omega),
$$

denotes the inner product. Note that the bilinear forms in (14) have clear physical interpretation [3] and due to continuity and ellipticity of some of them we can introduce the following energy norms:

$$\|\mathbf{v}\|_m^2 = m(\mathbf{v},\mathbf{v}), \qquad \|\mathbf{v}\|_c^2 = c(\mathbf{v},\mathbf{v}), \qquad \|\mathbf{v}\|_a^2 = a(\mathbf{v},\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V},$$

$$\|r\|_\chi^2 = \chi(r,r), \qquad \|r\|_z^2 = z(r,r) \qquad \forall r \in P,$$

$$\|\zeta\|_s^2 = s(\zeta,\zeta) \qquad \forall \zeta \in Z, \qquad \|\mathbf{y}\|_æ^2 = æ(\mathbf{y},\mathbf{y}) \qquad \forall \mathbf{y} \in \mathbf{H}.$$

3. Energy balance law. We assume the variational problem (13) admits a solution $\boldsymbol{\psi} = (\mathbf{u}, p, \theta, \mathbf{q})$. Then we substitute $\boldsymbol{\varphi} = (\mathbf{v}, r, \zeta, \mathbf{y}) = \boldsymbol{\psi}$ into (13) and after summation we obtain the following integral identity:

$$\frac{1}{2}\frac{d}{dt}[\|\mathbf{u}'(t)\|_m^2 + t_0\|\mathbf{q}(t)\|_æ^2 + \|\mathbf{u}(t)\|_c^2 + \|p(t)\|_\chi^2 + 2\pi(\theta(t),p(t)) + \|\theta(t)\|_s^2] +$$

$$+[\|\mathbf{u}'(t)\|_a^2 + \|p(t)\|_z^2 + \|\mathbf{q}(t)\|_æ^2] = \langle \mathbf{N}(t), \boldsymbol{\psi}(t)\rangle \qquad \forall t \in (0,T], \quad (15)$$

where

$$\langle \mathbf{N}, \boldsymbol{\varphi}\rangle := \langle \ell_\sigma, \mathbf{v}\rangle + \langle \ell_e, r\rangle + \langle \ell_\vartheta, \zeta\rangle \qquad \forall\, \boldsymbol{\varphi} = (\mathbf{v}, r, \zeta, \mathbf{y}) \in \boldsymbol{\Phi}.$$

Integrating (15) over any time interval $[0,t] \subseteq [0,T]$ we obtain a so-called energy balance equation for LS-thermopiezoelectricity:

$$\frac{1}{2}[\|\mathbf{u}'(t)\|_m^2 + t_0\|\mathbf{q}(t)\|_æ^2 + \|\mathbf{u}(t)\|_c^2 + \|p(t)\|_\chi^2 + 2\pi(\theta(t),p(t)) + \|\theta(t)\|_s^2] +$$

$$+\int_0^t [\|\mathbf{u}'(s)\|_a^2 + \|p(s)\|_z^2 + \|\mathbf{q}(s)\|_æ^2]\,ds =$$

$$= \frac{1}{2}[\|\mathbf{u}'(0)\|_m^2 + t_0\|\mathbf{q}(0)\|_æ^2 + \|\mathbf{u}(0)\|_c^2 + \|p(0)\|_\chi^2 +$$

$$+ 2\pi(\theta(0),p(0)) + \|\theta(0)\|_s^2] +$$

$$+\int_0^t \langle \mathbf{N}(s), \boldsymbol{\psi}(s)\rangle\,ds \qquad \forall t \in [0,T]. \quad (16)$$

Here

$$\frac{1}{2}[\|u'(t)\|_m^2 + t_0\|\mathbf{q}(t)\|\,|_æ^2] \equiv \mathcal{K}[\psi(t)]$$

is the kinetic energy,

$$\frac{1}{2}\|\boldsymbol{\psi}(t)\|_{\boldsymbol{\Phi}}^2 := \frac{1}{2}[\|\mathbf{u}(t)\|_c^2 + \|p(t)\|_\chi^2 + 2\pi(\theta(t),p(t)) + \|\theta(t)\|_s^2] \equiv \mathcal{E}[\boldsymbol{\psi}(t)]$$

is the potential energy, and

$$|\boldsymbol{\psi}(t)|_{\boldsymbol{\Phi}}^2 := \|\mathbf{u}'(t)\|_a^2 + \|p(t)\|_z^2 + \|\mathbf{q}(t)\|_æ^2$$

is the energy dissipation of the considered pyroelectric specimen.

In this terms the relation (16) can be rewritten in a shorter form:

$$\frac{1}{2}[\|\mathbf{u}'(t)\|_m^2 + t_0\|\mathbf{q}(t)\|_æ^2 + \|\boldsymbol{\psi}(t)\|_{\boldsymbol{\Phi}}^2] + \int_0^t |\boldsymbol{\psi}(s)|_{\boldsymbol{\Phi}}^2\,ds =$$

$$= \frac{1}{2}[\|\mathbf{u}'(0)\|_m^2 + t_0\|\mathbf{q}(0)\|_æ^2 + \|\boldsymbol{\psi}(0)\|_{\boldsymbol{\Phi}}^2] +$$

$$+\int_0^t \langle \mathbf{N}(s), \boldsymbol{\psi}(s)\rangle\,ds \qquad \forall t \in [0,T]. \quad (17)$$

4. A priori energy estimates for LS-thermopiezoelectricity. The energy identity (17) will be a basis for our future estimations. Here we use the technique similar to the one described in [15].

Taking into account initial conditions of the problem (13) and using Cauchy – Schwarz inequality we obtain

$$\|\mathbf{u}'(0)\|_m \le \|\mathbf{v}_0\|_m . \tag{18}$$

In similar way we receive

$$\|\boldsymbol{\psi}(0)\|_{\boldsymbol{\Phi}} \le \|\boldsymbol{\psi}_0\|_{\boldsymbol{\Phi}} , \qquad \|\mathbf{q}(0)\|_{æ} \le \|\mathbf{q}_0\|_{æ} , \tag{19}$$

where

$$\boldsymbol{\psi}_0 := (\mathbf{u}_0, p_0, \theta_0, \mathbf{q}_0) .$$

Again, using Cauchy – Schwarz inequality we obtain

$$\int_0^t \langle \mathbf{N}(s), \boldsymbol{\psi}(s) \rangle \, ds \le \frac{1}{2} \int_0^t [\|\mathbf{N}(s)\|_*^2 + \|\boldsymbol{\psi}(s)\|_{\boldsymbol{\Phi}}^2] \, ds \qquad \forall t \in [0, T] . \tag{20}$$

Substitution of the above estimates (18)–(20) into (17) gives the following a priori estimation:

$$\frac{1}{2}[\|\mathbf{u}'(t)\|_m^2 + t_0 \|\mathbf{q}(t)\|_{æ}^2 + \|\boldsymbol{\psi}(t)\|_{\boldsymbol{\Phi}}^2] + \int_0^t |\boldsymbol{\psi}(s)|_{\boldsymbol{\Phi}}^2 \, ds \le$$

$$\le \frac{1}{2}[\|\mathbf{v}_0\|_m^2 + t_0 \|\mathbf{q}_0\|_{æ}^2 + \|\boldsymbol{\psi}_0\|_{\boldsymbol{\Phi}}^2] + \frac{1}{2}\int_0^t \|\mathbf{N}(s)\|_*^2 \, ds +$$

$$+ \frac{1}{2}\int_0^t \|\boldsymbol{\psi}(s)\|_{\boldsymbol{\Phi}}^2 \, ds \qquad \forall t \in [0, T] ,$$

or more precisely

$$[\|\mathbf{u}'(t)\|_m^2 + t_0 \|\mathbf{q}(t)\|_{æ}^2 + \|\boldsymbol{\psi}(t)\|_{\boldsymbol{\Phi}}^2] + 2\int_0^t |\boldsymbol{\psi}(s)|_{\boldsymbol{\Phi}}^2 \, ds \le$$

$$\le [\|\mathbf{v}_0\|_m^2 + t_0 \|\mathbf{q}_0\|_{æ}^2 + \|\boldsymbol{\psi}_0\|_{\boldsymbol{\Phi}}^2] + \int_0^t \|\mathbf{N}(s)\|_*^2 \, ds +$$

$$+ \int_0^t \Big[\|\mathbf{u}'(s)\|_m^2 + t_0 \|\mathbf{q}(s)\|_{æ}^2 + \|\boldsymbol{\psi}(s)\|_{\boldsymbol{\Phi}}^2 +$$

$$+ 2\int_0^s |\boldsymbol{\psi}(\sigma)|_{\boldsymbol{\Phi}}^2 \, d\sigma \Big] ds \qquad \forall t \in [0, T] . \tag{21}$$

Now, applying the Gronwall's lemma to (21) we get

$$\|\mathbf{u}'(t)\|_m^2 + t_0 \|\mathbf{q}(t)\|_{æ}^2 + \|\boldsymbol{\psi}(t)\|_{\boldsymbol{\Phi}}^2 + 2\int_0^t |\boldsymbol{\psi}(s)|_{\boldsymbol{\Phi}}^2 \, ds \le C[\|\mathbf{v}_0\|_m^2 +$$

$$+ t_0 \|\mathbf{q}_0\|_{æ}^2 + \|\boldsymbol{\psi}_0\|_{\boldsymbol{\Phi}}^2 + \int_0^t \|\mathbf{N}(s)\|_*^2 \, ds] \qquad \forall t \in [0, T] , \tag{22}$$

where value $C = \text{const} > 0$ is independent from the variables of our interest.

Remark 1. The expression (22) shows that the most complete estimation of dynamic behavior of a pyroelectric specimen can be done by the following norm:

$$\left\| \boldsymbol{\psi}(t) \right\|_{\boldsymbol{\Phi}}^2 := \left\| \mathbf{u}'(t) \right\|_m^2 + t_0 \left\| \mathbf{q}(t) \right\|_{æ}^2 + \left\| \boldsymbol{\psi}(t) \right\|_{\boldsymbol{\Phi}}^2 + 2\int_0^t |\boldsymbol{\psi}(s)|_{\boldsymbol{\Phi}}^2 \, ds \quad \forall t \in [0,T].$$

Proposition 1. *The estimate* (22) *make sense if the input data of the LS-thermopiezoelectricity problem* (13) *satisfy the following regularity conditions:*

$$\mathbf{v}_0 \in [L^2(\Omega)]^d, \qquad \mathbf{N} \in L^2(0,T;\boldsymbol{\Phi}'),$$

$$\boldsymbol{\psi}_0 = (\mathbf{u}_0, p_0, \theta_0, \mathbf{q}_0) \in [H^1(\Omega)]^d \times L^2(\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^d. \tag{23}$$

Moreover, a solution $\boldsymbol{\psi} = (\mathbf{u}, p, \theta, \mathbf{q})$ *of the problem* (13), *if one exists*, *is characterized by the following inclusions*:

$$\mathbf{u}' \in L^\infty(0,T;[L^2(\Omega)]^d) \cap L^2(0,T;\mathbf{V}), \qquad \mathbf{u} \in L^\infty(0,T;\mathbf{V}) \cap L^2(0,T;\mathbf{V}),$$

$$p \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;P), \qquad \theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\Xi),$$

$$\mathbf{q} \in L^\infty(0,T;[L^2(\Omega)]^d) \cap L^2(0,T;\mathbf{H}), \tag{24}$$

and stability inequality

$$\left\| \boldsymbol{\psi}(t) \right\|_{\boldsymbol{\Phi}}^2 \le C\left[\left\| \mathbf{v}_0 \right\|_m^2 + t_0 \left\| \mathbf{q}_0 \right\|_{æ}^2 + \left\| \boldsymbol{\psi}_0 \right\|_{\boldsymbol{\Phi}}^2 + \int_0^t \left\| \mathbf{N}(s) \right\|_*^2 \, ds \right] \quad \forall t \in [0,T], \tag{25}$$

where the value $C = \text{const} > 0$ *is independent from the variables of our interest.*

Proposition 2. *A solution* $\boldsymbol{\psi} = (\mathbf{u}, p, \theta, \mathbf{q})$ *of the problem* (13), *if one exists, is unique.*

P r o o f. We prove this assertion by contradiction. That is, assume the contrary, namely, there exist different solutions $\boldsymbol{\psi}_1(t)$ and $\boldsymbol{\psi}_2(t)$ of the problem (13). Then their difference $\boldsymbol{\psi}(t) = \boldsymbol{\psi}_1(t) - \boldsymbol{\psi}_2(t) \neq 0$ satisfies the homogeneous equation of (13). Hence, by the Proposition 1 we have

$$\left\| \boldsymbol{\psi}(t) \right\|_{\boldsymbol{\Phi}}^2 \le 0 \qquad \forall t \in [0,T],$$

which contradicts the assumption $\boldsymbol{\psi}(t) \neq 0$. ♦

5. Finite element semidiscretization. In order to prove the existence of a solution $\boldsymbol{\psi} = (\mathbf{u}, p, \theta, \mathbf{q})$ of the problem (13) and to get an effective numerical algorithm for finding it, we use the finite element method.

To start discretization of the problem (13), we firstly triangulate the domain Ω. Let $\mathfrak{I}_h$ be a shape regular triangulation of the domain $\bar{\Omega}$ consisting of closed simplicial elements, $\mathfrak{I}_h = \{K\}$. We denote by $h_K = |K|^{1/d}$ the local mesh size for each element K, which is assumed to intersect at most one electrode surface Γ_p or Γ_e, and $h = \max_{K \in \mathfrak{I}_h} h_K$. Moreover, we suppose that we can generate a sequence of nested grids $\{\mathfrak{I}_h\}$ with $h \to 0$, for example, by the bisection method.

On the triangulation $\mathfrak{I}_h$ we define a piecewise polynomial finite element space $\boldsymbol{\Phi}_h := \mathbf{V}_h \times P_h \times Z_h \times \mathbf{H}_h \subset \boldsymbol{\Phi}$ with the components

$$\mathbf{V}_h = \left\{ \mathbf{v} \in \mathbf{V} \cap [C(\Omega)]^d : v|_K \in \mathrm{P}_m(K) \quad \forall K \in \mathfrak{I}_h \right\},$$

$$P_h = \left\{ r \in P \cap C(\Omega) : v|_K \in \mathrm{P}_m(K) \quad \forall K \in \mathfrak{I}_h \right\},$$

$$Z_h = \left\{ r \in \Xi \cap C(\Omega) : v|_K \in \mathrm{P}_m(K) \quad \forall K \in \mathfrak{I}_h \right\},$$

and

$$\mathbf{H}_h = \{\mathbf{y} \in \mathbf{H} : \mathbf{y}|_K \in \mathrm{RT}_{m-1}(K) \quad \forall K \in \Im_h\},$$

where $\mathrm{P}_m(K)$ is the space of polynomials of degree at most $m \ge 1$ defined on K, and $\mathrm{RT}_m(K)$ is the space of Raviart – Thomas polynomials

$$\mathrm{RT}_m(K) := [\mathrm{P}_m(K)]^d + \mathbf{x}\mathrm{P}_m(K), \qquad \mathbf{x} := (x_1, \dots, x_d)^\top$$

(see [4] for details). Here and below we assume that a set of spaces $\{\mathbf{\Phi}_h\}$ is dense in separable space $\mathbf{\Phi}$, $\mathbf{\Phi}_\Delta \subset \mathbf{\Phi}_h$, if $\Delta \ge h$.

Then for each $h > 0$ we define the semidiscrete finite element approximation $\boldsymbol{\psi}_h$ in such way:

given: $\quad h > 0, \quad \mathbf{N} = (\mathbf{l}_\sigma, \ell_e, \ell_\vartheta) \in \mathbf{V}' \times P' \times Z'$,

$$(\mathbf{v}_0, \mathbf{u}_0, p_0, \theta_0, \mathbf{q}_0) \in [L^2(\Omega)]^d \times \mathbf{V} \times L^2(\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^d,$$

find: $\quad \boldsymbol{\psi}_h = (\mathbf{u}_h, p_h, \theta_h, \mathbf{q}_h) \in L^2(0,T;\mathbf{\Phi}_h) \quad$ *such that*

$$\begin{aligned}
& m(\mathbf{u}_h''(t), \mathbf{v}) + a(\mathbf{u}_h'(t), \mathbf{v}) + c(\mathbf{u}_h(t), \mathbf{v}) - e(p_h(t), \mathbf{v}) - \\
& \qquad - \beta(\theta_h(t), \mathbf{v}) = \langle \mathbf{l}_\sigma(t), \mathbf{v} \rangle, \\
& \chi(p_h'(t), r) + z(p_h(t), r) + \pi(\theta_h'(t), r) + e(r, \mathbf{u}_h'(t)) = \langle \ell_e(t), r \rangle, \\
& s(\theta_h'(t), \zeta) + T_0^{-1}(\operatorname{div} \mathbf{q}_h(t), \zeta) + \pi(\zeta, p_h'(t)) + \beta(\zeta, \mathbf{u}_h'(t)) = \langle \ell_\vartheta(t), \zeta \rangle, \\
& t_0 æ(\mathbf{q}_h'(t), \mathbf{y}) - T_0^{-1}(\operatorname{div} \mathbf{y}, \theta_h(t)) + æ(\mathbf{q}_h(t), \mathbf{y}) = 0 \qquad \forall t \in (0, T], \\
& m(\mathbf{u}_h'(0) - \mathbf{v}_0, \mathbf{v}) = 0, \qquad c(\mathbf{u}_h(0) - \mathbf{u}_0, \mathbf{v}) = 0, \\
& \chi(p_h(0) - p_0, r) = 0, \qquad s(\theta_h(0) - \theta_0, \zeta) = 0, \\
& æ(\mathbf{q}_h(0) - \mathbf{q}_0, \mathbf{y}) = 0 \quad \forall \boldsymbol{\varphi} = (\mathbf{v}, r, \zeta, \mathbf{y}) \in \mathbf{\Phi}_h.
\end{aligned} \tag{26}$$

Since $\dim \mathbf{\Phi}_h < +\infty$, the problem (26) is the Cauchy problem for system of ordinary differential equations of the following kind:

$$\begin{aligned}
& MU''(t) + AU'(t) + CU(t) - E^\top P(t) - B^\top \theta(t) = L_\sigma(t), \\
& XP'(t) + \Pi^\top \theta'(t) + ZP(t) + EU'(t) = L_e(t), \\
& S\theta'(t) + \Pi P'(t) + W^\top F(t) + BU'(t) = L_\vartheta(t), \\
& K[F(t) + t_0 F'(t)] - W\theta(t) = 0 \qquad \forall\, t \in (0, T],
\end{aligned} \tag{27}$$

$$\begin{aligned}
& MU'(0) = V^0, \qquad CU(0) = U^0, \qquad XP(0) = P^0, \\
& S\theta(0) = \theta^0, \qquad KF(0) = F^0.
\end{aligned} \tag{28}$$

The initial value problem (27), (28) is nonsingular, therefore this problem is solvable. Moreover, taking into account the Proposition 1 we obtain the following

Proposition 3. *Let the input data of the LS-thermopiezoelectricity problem* (13) *satisfy the regularity conditions* (23).

Then for each $h > 0$ *there exists one and only one solution*

$$\boldsymbol{\psi}_h = (\mathbf{u}_h, p_h, \theta_h, \mathbf{q}_h)$$

of the problem (26) *such that*

$$\left\| \boldsymbol{\psi}_h(t) \right\|_{\boldsymbol{\Phi}}^2 \le C\left[\left\| \mathbf{v}_0 \right\|_m^2 + t_0 \left\| \mathbf{q}_0 \right\|_{æ}^2 + \left\| \boldsymbol{\psi}_0 \right\|_{\boldsymbol{\Phi}}^2 + \int_0^t \left\| \mathbf{N}(s) \right\|_*^2 \, ds \right] \quad \forall t \in [0, T], \quad (29)$$

where $C = \text{const} > 0$ *is independent from the variables of our interest.*

6. Existence of the solution for LS-thermopiezoelectricity problem. Now we are ready to formulate and prove the main result of this article.

Theorem 1. *Let us assume that the input data of LS-thermopiezoelectricity problem are characterized by regularity conditions* (23). *Then the variational problem* (13) *has a unique and stable solution* $\boldsymbol{\psi} = (\mathbf{u}, p, \theta, \mathbf{q})$ *which is characterized by the properties of regularity* (24) *and stability* (25).

P r o o f. We can conclude from (29) that when $h \to 0$, the sequence of semidiscrete approximations $\{\boldsymbol{\psi}_h\}$ ($\{\mathbf{u}_h'\}$ respectively) generates of a bounded subset in $L^2(0, T; \boldsymbol{\Phi})$ ($L^\infty(0, T; [L^2(\Omega)]^d)$ respectively). Therefore, we can select from the $\{\boldsymbol{\psi}_h\}$ ($\{\mathbf{u}_h'\}$ respectively) a subsequence $\{\boldsymbol{\psi}_\Delta\}$ ($\{\mathbf{u}_\Delta'\}$ respectively) such that $\boldsymbol{\psi}_\Delta$ ($\{\mathbf{u}_\Delta'\}$ respectively) $*$-weakly converges to $\boldsymbol{\psi}$ ($\mathbf{u}'$ respectively) in $L^2(0, T; \boldsymbol{\Phi})$ ($L^\infty(0, T; [L^2(\Omega)]^d)$ respectively).

We will now verify that $\boldsymbol{\psi}$ is a solution of the problem (13). We introduce the space of functions $\boldsymbol{\varphi}$ of the form

$$\mathbf{W}_\Delta := \left\{ \boldsymbol{\varphi} = (\mathbf{v}, r, \zeta, \mathbf{y}) \in C^1(0, T; \boldsymbol{\Phi}_\Delta) : \boldsymbol{\varphi}\big|_{t=T} = 0 \right\}.$$

Substituting $\boldsymbol{\varphi} \in \mathbf{W}_\Delta$ to the equations of (26) and integrating them on $(0, T)$ by parts, we obtain:

$$\int_0^T \left[-m(\mathbf{u}_h', \mathbf{v}') + a(\mathbf{u}_h', \mathbf{v}) + c(\mathbf{u}_h, \mathbf{v}) - e(p_h, \mathbf{v}) - \beta(\theta_h, \mathbf{v}) - \langle \mathbf{l}_\sigma, \mathbf{v} \rangle \right] dt =$$

$$= m(\mathbf{u}_h'(0), \mathbf{v}) = m(\mathbf{v}_0, \mathbf{v}),$$

$$\int_0^T \left[-\chi(p_h, r') + z(p_h, r) + \pi(\theta_h', r) + e(r, \mathbf{u}_h') - \langle \ell_e, r \rangle \right] dt =$$

$$= \chi(p_h(0), r) = \chi(p_0, r),$$

$$\int_0^T \left[-s(\theta_h, \zeta') + T_0^{-1}(\operatorname{div} \mathbf{q}_h, \zeta) + \pi(\zeta, p_h') + \beta(\zeta, \mathbf{u}_h') - \langle \ell_\vartheta, \zeta \rangle \right] dt =$$

$$= s(\theta_h(0), \zeta) = s(\theta_0, \zeta),$$

$$\int_0^T \left[-t_0 æ(\mathbf{q}_h, \mathbf{y}') - T_0^{-1}(\operatorname{div} \mathbf{y}, \theta_h) + æ(\mathbf{q}_h, \mathbf{y}) \right] dt = t_0 æ(\mathbf{q}_h(0), \mathbf{y}) =$$

$$= t_0 æ(\mathbf{q}_0, \mathbf{y}) \qquad \forall \boldsymbol{\varphi} = (\mathbf{v}, r, \zeta, \mathbf{y}) \in \mathbf{W}_\Delta.$$

Passing $h \to 0$ and again using integration by parts we get the following system of equations:

$$\int_0^T \left[m(\mathbf{u}'', \mathbf{v}) + a(\mathbf{u}', \mathbf{v}) + c(\mathbf{u}, \mathbf{v}) - e(p, \mathbf{v}) - \beta(\theta, \mathbf{v}) - \langle \mathbf{l}_\sigma, \mathbf{v}\rangle\right] dt =$$

$$= -\, m(\mathbf{u}'(0) - \mathbf{v}_0, \mathbf{v}),$$

$$\int_0^T \left[\chi(p', r) + z(p, r) + \pi(\theta', r) + e(r, \mathbf{u}') - \langle \ell_e, r\rangle\right] dt = -\,\chi(p(0) - p_0, r),$$

$$\int_0^T \left[s(\theta', \zeta) + T_0^{-1}(\operatorname{div}\mathbf{q}, \zeta) + \pi(\zeta, p') + \beta(\zeta, \mathbf{u}') - \langle \ell_\vartheta(t), \zeta\rangle\right] dt =$$

$$= -\, s(\theta(0) - \theta_0, \zeta),$$

$$\int_0^T \left[t_0 æ(\mathbf{q}', \mathbf{y}) - T_0^{-1}(\operatorname{div}\mathbf{y}, \theta) + æ(\mathbf{q}, \mathbf{y})\right] dt =$$

$$= -\, t_0 æ(\mathbf{q}(0) - \mathbf{q}_0, \mathbf{y}) \qquad \forall \boldsymbol{\varphi} = (\mathbf{v}, r, \zeta, \mathbf{y}) \in \mathbf{W}_\Delta. \tag{30}$$

Since set $\boldsymbol{\Phi}_\Delta$ is dense in space $\boldsymbol{\Phi}$, equations (30) are true for each $\boldsymbol{\varphi} \in C^1([0, T]; \boldsymbol{\Phi})$. Therefore, we can conclude, that

$$m(\mathbf{u}''(t), \mathbf{v}) + a(\mathbf{u}'(t), \mathbf{v}) + c(\mathbf{u}(t), \mathbf{v}) - e(p(t), \mathbf{v}) - \beta(\theta(t), \mathbf{v}) = \langle \mathbf{l}_\sigma(t), \mathbf{v}\rangle,$$

$$\chi(p'(t), r) + z(p(t), r) + \pi(\theta'(t), r) + e(r, \mathbf{u}'(t)) = \langle \ell_e(t), r\rangle,$$

$$s(\theta'(t), \zeta) + T_0^{-1}(\operatorname{div}\mathbf{q}(t), \zeta) + \pi(\zeta, p'(t)) + \beta(\zeta, \mathbf{u}'(t)) = \langle \ell_\vartheta(t), \zeta\rangle,$$

$$t_0 æ(\mathbf{q}'(t), \mathbf{y}) - T_0^{-1}(\operatorname{div}\mathbf{y}, \theta(t)) + æ(\mathbf{q}(t), \mathbf{y}) = 0 \qquad \forall t \in (0, T],$$

$$m(\mathbf{u}'(0) - \mathbf{v}_0, \mathbf{v}) = 0, \qquad \chi(p(0) - p_0, r) = 0, \qquad s(\theta(0) - \theta_0, \zeta) = 0,$$

$$æ(\mathbf{q}(0) - \mathbf{q}_0, \mathbf{y}) = 0 \qquad \forall \boldsymbol{\varphi} = (\mathbf{v}, r, \zeta, \mathbf{y}) \in \boldsymbol{\Phi}.$$

Finally, from the unused initial condition of (26) we get

$$c(\mathbf{u}_0, \mathbf{v}) = c(\mathbf{u}_\Delta(0), \mathbf{v}) \to c(\mathbf{u}(0), \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}.$$

Conclusions. The initial boundary value Lord – Shulman thermopiezoelectricity problem has been converted to the corresponding variational problem. Starting with energy balance law, we has been able to establish the sufficient conditions for regularity of the problem of input data, which guarantee the uniqueness and continuous dependence of the solution of the variational problem. We have also presented the constructive proof of existence of solution by means of Galerkin semidiscretization by spatial variables. Such approach allows us to construct a reasonable procedure for determination the approximate solution of the problem with use of the finite element method. The complete numerical scheme for solving Lord – Shulman thermopiezoelectricity variational problem can be obtained by complementing Galerkin semidiscretization by spatial variables with discretization in time. For example, one-step recurrent scheme (like in our previous works [3, 6]) can be used for this purpose.

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КОРЕКТНІСТЬ ВАРІАЦІЙНОЇ ЗАДАЧІ ТЕРМОП'ЄЗОЕЛЕКТРИКИ ЛОРДА – ШУЛЬМАНА

На основі початково-крайової задачі термоп'єзоелектрики Лорда – Шульмана сформульовано відповідну їй варіаційну задачу в термінах вектора пружних зміщень, електричного потенціалу, приросту температури та вектора теплових потоків. З використанням енергетичного рівняння варіаційної задачі встановлено достатні умови регулярності вхідних даних задачі, а також доведено єдиність її розв'язку. У доведенні існування узагальненого розв'язку використано напівдискретизацію Гальоркіна за просторовими змінними і показано, що границя послідовності її наближень є розв'язком варіаційної задачі термоп'єзоелектрики Лорда – Шульмана, що дає можливість побудувати обґрунтовану процедуру обчислення апроксимації розв'язку цієї задачі.

КОРРЕКТНОСТЬ ВАРИАЦИОННОЙ ЗАДАЧИ ТЕРМОПЬЕЗОЭЛЕКТРИЧЕСТВА ЛОРДА – ШУЛЬМАНА

На основании начально-краевой задачи термопьезоэлектричества Лорда – Шульмана сформулирована соответствующая ей вариационная задача в терминах вектора упругих смещений, электрического потенциала, приращения температуры и вектора тепловых потоков. С использованием энергетического уравнения вариационной задачи установлены достаточные условия регулярности исходных данных задачи, а также доказана единственность ее решения. В доказательстве существования обобщенного решения использована полудискретизация Галеркина по пространственным переменным и показано, что предел последовательности ее приближений является решением вариационной задачи термопьезоэлектричества Лорда – Шульмана, что дает возможность построить обоснованную процедуру вычисления аппроксимации решения этой задачи.

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