УДК 539.3

L. A. Aghalovyan[1], L. G. Ghulghazaryan[1,2], J. D. Kaplunov[3], D. A. Prikazchikov[3✉]

3D DYNAMIC ANALYSIS OF LAYERED ELASTIC SHELLS

Three-dimensional dynamic problem for a layered orthotropic elastic shell with free upper face is considered. The interfaces between the layers are assumed to be in perfect contact and the displacements on one of the interfaces are prescribed. A long-wave asymptotic solution is constructed and the thickness resonances are determined. The obtained results can be applied in evaluation of certain parameters of earthquakes.

Key words: *3D elastodynamics, layered shell, asymptotic method.*

Introduction. Mathematical modeling of thin elastic multi-layered solids with various boundary conditions on the faces is of significant importance for many applications. Among the latter we mention the theories related to earthquake prediction [19, 22, 25], relying on the data of displacements measured at certain points of the region under investigation. It is emphasized that restoration of associated dynamic parameters of the stress-strain state using the measured discrete data and taking into account possible curvature of the layers is of crucial importance for seismological theories.

Problems of mechanics of multi-layered elastic plates and shells with non-classical boundary conditions, i.e. conditions imposed not only on the stress tensor component, have been studied in various publications, see e.g. a monograph [6], as well as journal papers including both analyses of free [1, 7, 8] and forced vibrations [2, 9, 14]. We also mention important contributions to the study of free vibrations of single-layer plates and shells in the case where one or both faces is fixed [15, 17, 20, 23, 26]. The associated long-wave high-frequency motions, investigated in these papers, have also been thoroughly studied in the case of classical boundary conditions (formulated in terms of stresses), see [3, 5, 21].

The current paper is devoted to the further developing results obtained in the above-mentioned works. The approach relies on the asymptotic method, widely used in statics and dynamics of thin-walled elastic structures, see e.g. [4, 16], as well as recent monographs [10, 25], and publications [11−13, 18, 24] to name a few, accounting for the effects of pre-stress, nonlocality, high contrast, and also in contact problems for coated solids.

The asymptotic technique employed in this paper starts with a scaling typical for non-classical face boundary conditions. 3D dynamic problems for two- and three-layered elastic orthotropic shells with a traction-free upper face are considered. In addition, it is assumed that the displacements are prescribed on one of the contact surfaces between the layers; in particular, for a two-layered shell they are imposed on the contact surface between the first and second layers, whereas for a three-layered shell both situations are considered, namely when the displacements are given on the contact surface between the first and second layers, and between second and third layers. The layers are supposed to be in perfect contact. Within the current consideration, it is also assumed that the wave length exceeds substantially the thickness, thus providing a natural geometrical small parameter. The frequency range covers the thickness resonances. The parameters of the stress-strain state are expanded into asymptotic series. The iterative formulae for the coefficients of these series are derived. It is remarked that in the case of the excitation frequency coinciding with one of the thickness resonant frequencies, further investigation is required, according to the procedure

✉ d.prikazchikov@keele.ac.uk

presented in [15] etc. Explicit asymptotic results for the displacements are obtained, which could be useful for estimates of certain parameters of earthquakes.

1. Forced vibrations of a two-layered elastic shell. Consider a two-layered elastic orthotropic shell (Fig. 1), occupying the domain $D = \{\alpha, \beta, \gamma;\ \alpha, \beta \in D_0,\ 0 \le \gamma \le h_1 + h_2\}$, where D_0 is the face surface of the first layer, α, β are lines of curvature of the surface D_0, γ is rectangular axis directed downward to the surface D_0. The non-trivial solutions are sought for the problem of elastodynamics in the given triorthogonal coordinate system. The shell is under non-classical boundary conditions (which will be formulated below). In order to reduce the length of algebraic computations, the analysis below is presented in terms of the components τ_{ij} of non-symmetric stress tensor, see [4, 25].

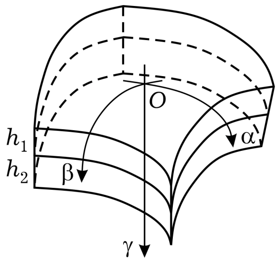

Fig. 1

The formulation of the problem includes:

– *equations of motion*

$$\frac{1}{AB}\frac{\partial}{\partial\alpha}\left(B\tau_{\alpha\alpha}^{(j)}\right) - k_\beta \tau_{\beta\beta}^{(j)} + \frac{1}{AB}\frac{\partial}{\partial\beta}\left(A\tau_{\beta\alpha}^{(j)}\right) + k_\alpha \tau_{\alpha\beta}^{(j)} + \left(1+\frac{\gamma}{R_1}\right)\frac{\partial \tau_{\alpha\psi}^{(j)}}{\partial\gamma} +$$

$$+\frac{2\tau_{\alpha\gamma}^{(j)}}{R_1} = \rho^{(j)}\left(1+\frac{\gamma}{R_1}\right)\left(1+\frac{\gamma}{R_2}\right)\frac{\partial^2 U^{(j)}}{\partial t^2},$$

$$(A \leftrightarrow B;\ \ \alpha \leftrightarrow \beta;\ \ R_1, R_2;\ \ U, V), \qquad j = \mathbf{I}, \mathbf{II},$$

$$\frac{\partial \tau_{\gamma\gamma}^{(j)}}{\partial\gamma} - \left(\frac{\tau_{\alpha\alpha}^{(j)}}{R_1} + \frac{\tau_{\beta\beta}^{(j)}}{R_2}\right) + \frac{1}{A}\frac{\partial \tau_{\alpha\gamma}^{(j)}}{\partial\alpha} + \frac{1}{B}\frac{\partial \tau_{\beta\gamma}^{(j)}}{\partial\beta} +$$

$$+ k_\beta \tau_{\alpha\gamma}^{(j)} + k_\alpha \tau_{\beta\gamma}^{(j)} = \rho^{(j)}\left(1+\frac{\gamma}{R_1}\right)\left(1+\frac{\gamma}{R_2}\right)\frac{\partial^2 W^{(j)}}{\partial t^2}, \qquad (1)$$

$$\left(1+\frac{\gamma}{R_1}\right)\tau_{\alpha\beta}^{(j)} = \left(1+\frac{\gamma}{R_2}\right)\tau_{\beta\alpha}^{(j)} \qquad \text{(symmetry relation)};$$

– *constitutive relations for an orthotropic elastic solid*:

$$\left(1+\frac{\gamma}{R_2}\right)\left(\frac{1}{A}\frac{\partial U^{(j)}}{\partial\alpha} + k_\alpha V^{(j)} + \frac{W^{(j)}}{R_1}\right) =$$

$$= \left(1+\frac{\gamma}{R_1}\right)a_{11}^{(j)}\tau_{\alpha\alpha}^{(j)} + \left(1+\frac{\gamma}{R_2}\right)a_{12}^{(j)}\tau_{\beta\beta}^{(j)} + a_{13}^{(j)}\tau_{\gamma\gamma}^{(j)},$$

$$(A, B;\ \ \alpha \leftrightarrow \beta;\ \ R_1 \leftrightarrow R_2;\ \ U \leftrightarrow V;\ \ a_{11}, a_{22};\ \ a_{13}, a_{23}),$$

$$\left[1 + \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{\gamma^2}{R_1 R_2}\right]\frac{\partial W^{(j)}}{\partial\gamma} =$$

$$= \left(1+\frac{\gamma}{R_1}\right)a_{13}^{(j)}\tau_{\alpha\alpha}^{(j)} + \left(1+\frac{\gamma}{R_2}\right)a_{23}^{(j)}\tau_{\beta\beta}^{(j)} + a_{33}^{(j)}\tau_{\gamma\gamma}^{(j)},$$

$$\left(1+\frac{\gamma}{R_1}\right)\left(\frac{1}{B}\frac{\partial U^{(j)}}{\partial\beta} - k_\beta V^{(j)}\right) + \left(1+\frac{\gamma}{R_2}\right)\left(\frac{1}{A}\frac{\partial V^{(j)}}{\partial\alpha} - k_\alpha U^{(j)}\right) =$$

$$= \left(1 + \frac{\gamma}{R_1}\right) a_{66}^{(j)} \tau_{\alpha\beta}^{(j)} ,$$

$$\left[1 + \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{\gamma^2}{R_1 R_2}\right] \frac{\partial U^{(j)}}{\partial \gamma} -$$

$$- \left(1 + \frac{\gamma}{R_2}\right) \frac{U^{(j)}}{R_1} + \frac{1}{A}\left(1 + \frac{\gamma}{R_2}\right) \frac{\partial W^{(j)}}{\partial \alpha} = \left(1 + \frac{\gamma}{R_1}\right) a_{55}^{(j)} \tau_{\alpha\gamma}^{(j)} ,$$

$$(A, B; \ \alpha, \beta; \ R_1 \leftrightarrow R_2; \ U, V; \ a_{55}, a_{44}), \qquad (2)$$

where k_α, k_β denote geodesic curvatures, A, B are coefficients of the first quadratic form, R_1, R_2 are main curvature radii of surface D_0, $\rho^{(j)}$ are mass volume densities of the layers, $a_{ik}^{(j)}$ are elastic constants, $a_{ik}^{(j)} = a_{ki}^{(j)}$, and j is number of the layer.

Let the upper face $\gamma = 0$ be traction-free:

$$\tau_{\alpha\gamma}^{\mathbf{I}}(\alpha, \beta, 0, t) = 0, \quad \tau_{\beta\gamma}^{\mathbf{I}}(\alpha, \beta, 0, t) = 0, \quad \tau_{\gamma\gamma}^{\mathbf{I}}(\alpha, \beta, 0, t) = 0 , \qquad (3)$$

and assume that on the contact surface between the first and second layers the prescribed displacements are given by:

$$U^{\mathbf{I}}(\alpha, \beta, h_1, t) = U^{\mathbf{II}}(\alpha, \beta, h_1, t) = U^{+}(\alpha, \beta) \exp(i\Omega t), \quad (U, V, W) , \qquad (4)$$

where Ω is a given excitation frequency of forced vibrations. Perfect contact between the layers is assumed:

$$\tau_{\alpha\gamma}^{\mathbf{I}}(\alpha, \beta, h_1, t) = \tau_{\alpha\gamma}^{\mathbf{II}}(\alpha, \beta, h_1, t), \quad (\tau_{\alpha\gamma}, \tau_{\beta\gamma}, \tau_{\gamma\gamma}) , \qquad (5)$$

$$U^{\mathbf{I}}(\alpha, \beta, h_1, t) = U^{\mathbf{II}}(\alpha, \beta, h_1, t), \quad (U, V, W). \qquad (6)$$

If the displacements were prescribed on the traction-free surface, then in addition to conditions (3) the equation

$$U^{\mathbf{I}}(\alpha, \beta, 0, t) = U^{+}(\alpha, \beta) \exp(i\Omega t) , \quad (U, V, W) ,$$

should hold, i.e. at $\gamma = 0$ there would be six conditions, whereas in classical elasticity there are usually only three conditions. Also, at $\gamma = h_1$, conditions (5), (6) should hold, as well as three more conditions (4), so once again we have non-classical boundary conditions. As demonstrated in [7], the solution of the formulated boundary value problem always exists, moreover, it coincides with the solution of some problem with classical boundary conditions.

Let us introduce the scaling

$$\alpha = R\xi, \quad \beta = R\eta, \quad \gamma = \varepsilon R\zeta = h\zeta ,$$

$$U = Ru, \quad V = Rv, \quad W = Rw, \quad \tau_{mk}^{(j)} = \mu \tilde{\tau}_{mk}^{(j)}, \quad \rho^{(j)} = \rho\, \tilde{\rho}^{(j)} ,$$

where R is a typical linear size of the shell, total thickness $h = h_1 + h_2$ (e.g. the least of the radii of curvature and associated with linear sizes of surface D_0, $h \ll R$), μ and ρ are typical values of elastic moduli and density, respectively, and

$$\varepsilon = h/R$$

as a small geometrical parameter.

The solutions are sought for in the form:

$$Q_{\alpha\beta}^{(j)} = Q_{mk}^{(j)}(\xi, \eta, \zeta) \exp(i\Omega t) \ \ (\alpha, \beta, \gamma), \quad m, k = 1, 2, 3, \ \ j = \mathbf{I}, \mathbf{II} , \qquad (7)$$

where $Q_{\alpha\beta}^{(j)}$ denotes any of the stress or displacement components. As a result, we arrive at a singularly perturbed system with respect to $Q_{mk}^{(j)}$ with a small parameter ε. The stresses and displacements are now represented in asymptotic form as

$$\tilde{\tau}_{mk}^{(j)}(\xi,\eta,\zeta) = \varepsilon^{-1+s}\tilde{\tau}_{mk}^{(j,s)}(\xi,\eta,\zeta), \quad m,k = 1,2,3, \quad s = 0,\dots,N, \quad j = \mathbf{I},\mathbf{II},$$

$$\left(u^{(j)}(\xi,\eta,\zeta), v^{(j)}(\xi,\eta,\zeta), w^{(j)}(\xi,\eta,\zeta)\right) =$$

$$= \varepsilon^{s}\left(u^{(j,s)}(\xi,\eta,\zeta), v^{(j,s)}(\xi,\eta,\zeta), w^{(j,s)}(\xi,\eta,\zeta)\right). \tag{8}$$

As follows from (8), there is a significant distinction from the conventional classical problem, related to relative orders, in particular, all stress components are asymptotically of the same order, and so are all displacements, therefore the traditional assumptions of plate and shell theory are invalid.

Substituting (8) into the dimensionless forms of the governing equations (1), (2), we arrive at a system of the equations for determining the unknown coefficients of expansion $Q_{mk}^{(j,s)}$ in the form

$$\tilde{\tau}_{12}^{(j,s)} = P_{1\tau}^{(j,s-1)}, \quad \tilde{\tau}_{21}^{(j,s)} = P_{1\tau}^{(j,s-1)} - r_2\zeta\tilde{\tau}_{21}^{(j,s-1)} + r_1\zeta\tilde{\tau}_{12}^{(j,s-1)},$$

$$e_1^{(j,s)} = P_{2\tau}^{(j,s-1)}, \quad e_2^{(j,s)} = P_{3\tau}^{(j,s-1)},$$

$$\frac{\partial\tilde{\tau}_{13}^{(j,s)}}{\partial\zeta} + \tilde{\Omega}_*^2\tilde{\rho}^{(j)}u^{(j,s)} = P_{6\tau}^{(j,s-1)}, \quad \tilde{\Omega}_*^2 = \frac{\rho}{\mu}h^2\Omega^2, \quad \tilde{a}_{mk}^{(j)} = \mu a_{mk}^{(j)},$$

$$(13, 23, 33; \quad u, v, w; \quad 6\tau, 5\tau, 4\tau),$$

$$\frac{\partial u^{(j,s)}}{\partial\zeta} - \tilde{a}_{55}^{(j)}\tilde{\tau}_{13}^{(j,s)} = P_u^{(j,s-1)}, \quad \frac{\partial v^{(j,s)}}{\partial\zeta} - \tilde{a}_{44}^{(j)}\tilde{\tau}_{23}^{(j,s)} = P_v^{(j,s-1)},$$

$$\frac{\partial w^{(j,s)}}{\partial\zeta} - e_3^{(j,s)} = P_w^{(j,s-1)}, \tag{9}$$

where

$$P_{1\tau}^{(j,s-1)} = \frac{1}{\tilde{a}_{66}^{(j)}}\left[\frac{1}{B}\frac{\partial u^{(j,s-1)}}{\partial\eta} - k_\beta R v^{(j,s-1)} + \right.$$

$$+ r_1\zeta\left(\frac{1}{B}\frac{\partial u^{(j,s-2)}}{\partial\eta} - k_\beta R v^{(j,s-2)}\right) + \frac{1}{A}\frac{\partial v^{(j,s-1)}}{\partial\xi} - k_\alpha R u^{(j,s-1)} +$$

$$\left. + r_2\zeta\left(\frac{1}{A}\frac{\partial v^{(j,s-2)}}{\partial\xi} - k_\alpha R u^{(j,s-2)}\right) - r_1\zeta\,\tilde{a}_{66}^{(j)}\tilde{\tau}_{12}^{(j,s-1)}\right],$$

$$P_{2\tau}^{(j,s-1)} = \frac{1}{A}\frac{\partial u^{(j,s-1)}}{\partial\xi} + k_\alpha R v^{(j,s-1)} + r_1 w^{(j,s-1)} - r_1\zeta\tilde{a}_{11}^{(j)}\tilde{\tau}_{11}^{(j,s-1)} +$$

$$+ r_2\zeta\left(\frac{1}{A}\frac{\partial u^{(j,s-2)}}{\partial\xi} + k_\alpha R v^{(j,s-2)} + r_1 w^{(j,s-2)}\right) -$$

$$- r_2\zeta\tilde{a}_{12}^{(j)}\tilde{\tau}_{22}^{(j,s-1)},$$

$$(2\tau, 3\tau; \quad A, B; \quad \alpha, \beta; \quad r_1 \leftrightarrow r_2; \quad \xi, \eta; \quad u \leftrightarrow v; \quad \tau_{11} \leftrightarrow \tau_{22}; \quad a_{11}, a_{22});$$

$$P_{4\tau}^{(j,s-1)} = r_1\tilde{\tau}_{11}^{(j,s-1)} + r_2\tilde{\tau}_{22}^{(j,s-1)} - \frac{1}{A}\frac{\partial\tilde{\tau}_{13}^{(j,s-1)}}{\partial\xi} - \frac{1}{B}\frac{\partial\tilde{\tau}_{23}^{(j,s-1)}}{\partial\eta} - k_\beta R\tilde{\tau}_{13}^{(j,s-1)} -$$

$$- k_\alpha R\tilde{\tau}_{23}^{(j,s-1)} - \tilde{\rho}^{(j)}(r_1 + r_2)\zeta\tilde{\Omega}_*^2 w^{(j,s-1)} - \tilde{\rho}^{(j)} r_1 r_2 \zeta^2 \tilde{\Omega}_*^2 w^{(j,s-2)},$$

$$P_{5\tau}^{(j,s-1)} = -\frac{1}{AB}\frac{\partial}{\partial\eta}\left(A\tilde{\tau}_{22}^{(j,s-1)}\right) + k_\alpha R\tilde{\tau}_{11}^{(j,s-1)} - \frac{1}{AB}\frac{\partial}{\partial\xi}\left(B\tilde{\tau}_{12}^{(j,s-1)}\right) -$$

$$- k_\beta R\tilde{\tau}_{21}^{(j,s-1)} - r_2\zeta\frac{\partial\tilde{\tau}_{23}^{(j,s-1)}}{\partial\zeta} - 2r_2\tilde{\tau}_{23}^{(j,s-1)} -$$

$$- (r_1 + r_2)\zeta\,\tilde{\rho}^{(j)}\tilde{\Omega}_*^2 v^{(j,s-1)} - \tilde{\rho}^{(j)} r_1 r_2 \zeta^2 \tilde{\Omega}_*^2 v^{(j,s-2)},$$

$$(5\tau, 6\tau;\ \ A \leftrightarrow B;\ \ \alpha \leftrightarrow \beta;\ \ r_2, r_1;\ \ v, u;\ \ \xi \leftrightarrow \eta;$$

$$\tau_{11} \leftrightarrow \tau_{22};\ \ \tau_{12} \leftrightarrow \tau_{21};\ \ \tau_{23}, \tau_{13});$$

$$P_u^{(j,s-1)} = -\zeta(r_1 + r_2)\frac{\partial u^{(j,s-1)}}{\partial\zeta} - \zeta^2 r_1 r_2 \frac{\partial u^{(j,s-2)}}{\partial\zeta} + r_1 u^{(j,s-1)} +$$

$$+ \zeta r_1 r_2 u^{(j,s-2)} - \frac{1}{A}\frac{\partial w^{(j,s-1)}}{\partial\xi} - \frac{r_2\zeta}{A}\frac{\partial w^{(j,s-2)}}{\partial\xi} + r_1\zeta\tilde{a}_{55}^{(j)}\tilde{\tau}_{13}^{(j,s-1)},$$

$$(u, v;\ \ A, B;\ \ r_1 \leftrightarrow r_2;\ \ \xi, \eta;\ \ \tau_{13}, \tau_{23};\ \ a_{55}, a_{44});$$

$$P_w^{(j,s-1)} = -\zeta(r_1 + r_2)\frac{\partial w^{(j,s-1)}}{\partial\zeta} - \zeta^2 r_1 r_2 \frac{\partial w^{(j,s-2)}}{\partial\zeta} +$$

$$+ r_1\zeta\tilde{a}_{13}^{(j)}\tilde{\tau}_{11}^{(j,s-1)} + r_2\zeta\tilde{a}_{23}^{(j)}\tilde{\tau}_{22}^{(j,s-1)},$$

$$r_1 = \frac{R}{R_1}, \quad r_2 = \frac{R}{R_2},$$

$$e_i^{(j,s)} = \tilde{a}_{i1}^{(j)}\tilde{\tau}_{11}^{(j,s)} + \tilde{a}_{i2}^{(j)}\tilde{\tau}_{22}^{(j,s)} + \tilde{a}_{i3}^{(j)}\tilde{\tau}_{33}^{(j,s)}, \quad i = 1, 2, 3, \quad j = \mathbf{I}, \mathbf{II}. \tag{10}$$

In view of (9), stress tensor components may be expressed in terms of $u^{(j,s)}, v^{(j,s)}, w^{(j,s)}$ as

$$\tilde{\tau}_{13}^{(j,s)} = \frac{1}{\tilde{a}_{55}^{(j)}}\left[\frac{\partial u^{(j,s)}}{\partial\zeta} - P_u^{(j,s-1)}\right], \quad \tilde{\tau}_{23}^{(j,s)} = \frac{1}{\tilde{a}_{44}^{(j)}}\left[\frac{\partial v^{(j,s)}}{\partial\zeta} - P_v^{(j,s-1)}\right],$$

$$\tilde{\tau}_{12}^{(j,s)} = P_{1\tau}^{(j,s-1)}, \quad \tilde{\tau}_{21}^{(s)} = P_{1\tau}^{(j,s-1)} - r_2\zeta\tilde{\tau}_{21}^{(j,s-1)} + r_1\zeta\tilde{\tau}_{12}^{(j,s-1)},$$

$$\tilde{\tau}_{11}^{(j,s)} = \frac{1}{\Delta^{(j)}}\left[\Delta_2^{(j)}\frac{\partial w^{(j,s)}}{\partial\zeta} + \Delta_{23}^{(j)} P_{2\tau}^{(j,s-1)} + \Delta_1^{(j)} P_{3\tau}^{(j,s-1)} - \Delta_2^{(j)} P_w^{(j,s-1)}\right],$$

$$(11, 22, 33;\ \ \Delta_2, \Delta_3, \Delta_{12};\ \ \Delta_{23}, \Delta_1, \Delta_2;\ \ \Delta_1, \Delta_{13}, \Delta_3), \tag{11}$$

where

$$\Delta_1^{(j)} = \tilde{a}_{13}^{(j)}\tilde{a}_{23}^{(j)} - \tilde{a}_{33}^{(j)}\tilde{a}_{12}^{(j)}, \quad \Delta_2^{(j)} = \tilde{a}_{12}^{(j)}\tilde{a}_{23}^{(j)} - \tilde{a}_{22}^{(j)}\tilde{a}_{13}^{(j)},$$

$$\Delta_3^{(j)} = \tilde{a}_{13}^{(j)}\tilde{a}_{12}^{(j)} - \tilde{a}_{11}^{(j)}\tilde{a}_{23}^{(j)}, \quad \Delta^{(j)} = \tilde{a}_{11}^{(j)}\Delta_{23}^{(j)} + \tilde{a}_{13}^{(j)}\Delta_2^{(j)} + \tilde{a}_{12}^{(j)}\Delta_1^{(j)},$$

$$\Delta_{ik}^{(j)} = \tilde{a}_{ii}^{(j)}\tilde{a}_{kk}^{(j)} - (\tilde{a}_{ik}^{(j)})^2, \quad i, k = 1, 2, 3. \tag{12}$$

According to (9), displacement components satisfy the following equations:

$$\frac{\partial^2 u^{(j,s)}}{\partial \zeta^2} + \tilde{a}_{55}^{(j)} \tilde{\Omega}_*^2 \tilde{\rho}^{(j)} u^{(j,s)} = \tilde{a}_{55}^{(j)} P_{6\tau}^{(j,s-1)} + \frac{\partial P_u^{(j,s-1)}}{\partial \zeta},$$

$$(u, v;\ \ a_{55}, a_{44};\ \ 6\tau, 5\tau),$$

$$\frac{\partial^2 w^{(j,s)}}{\partial \zeta^2} + \frac{\Delta^{(j)}}{\Delta_{12}^{(j)}} \tilde{\Omega}_*^2 \tilde{\rho}^{(j)} w^{(j,s)} = F_w^{(j,s-1)},$$

$$F_w^{(j,s-1)} = \frac{1}{\Delta_{12}^{(j)}} \left[\Delta^{(j)} P_{4\tau}^{(j,s-1)} - \Delta_2^{(j)} \frac{\partial P_{2\tau}^{(j,s-1)}}{\partial \zeta} - \Delta_3^{(j)} \frac{\partial P_{3\tau}^{(j,s-1)}}{\partial \zeta} + \right.$$

$$\left. + \Delta_{12}^{(j)} \frac{\partial P_w^{(j,s-1)}}{\partial \zeta} \right], \quad j = \mathbf{I}, \mathbf{II}. \tag{13}$$

Solutions of (13) are given by

$$u^{(j,s)}(\xi, \eta, \zeta) = C_1^{(j,s)}(\xi, \eta) \sin \chi^{(j,u)} \zeta + C_2^{(j,s)}(\xi, \eta) \cos \chi^{(j,u)} \zeta + \overline{u}^{(j,s)}(\xi, \eta, \zeta),$$

$$(u, v, w;\ \ 1, 3, 5;\ \ 2, 4, 6), \quad j = \mathbf{I}, \mathbf{II}, \tag{14}$$

where

$$\chi^{(j,u)} = \sqrt{\tilde{a}_{55}^{(j)} \tilde{\rho}^{(j)}}\, \tilde{\Omega}_*, \quad \chi^{(j,v)} = \sqrt{\tilde{a}_{44}^{(j)} \tilde{\rho}^{(j)}}\, \tilde{\Omega}_*, \quad \chi^{(j,w)} = \sqrt{\frac{\Delta^{(j)} \rho^{(j)}}{\Delta_{12}^{(j)}}}\, \tilde{\Omega}_*,$$

with $\overline{u}^{(j,s)}, \overline{v}^{(j,s)}, \overline{w}^{(j,s)}$ being particular solutions of equations (13).

Satisfying the conditions (3)–(6), we obtain algebraic systems with respect to the unknowns $C_i^{(j,s)}$, $i = 1, \ldots, 6$, $j = \mathbf{I}, \mathbf{II}$, that have finite solutions provided

$$\cos \chi^{(\mathbf{I},u)} \zeta_1 \neq 0, \quad \zeta_1 = h_1 / h, \quad (u, v, w). \tag{15}$$

Solving these systems, we get the displacements

– for the first layer:

$$u^{(\mathbf{I},s)}(\xi, \eta, \zeta) = \frac{B_2^{(u,s)}(\xi, \eta) \cos \chi^{(\mathbf{I},u)} \zeta + B_1^{(u,s)}(\xi, \eta) \sin \chi^{(\mathbf{I},u)} (\zeta - \zeta_1)}{\cos \chi^{(\mathbf{I},u)} \zeta_1} +$$

$$+ \overline{u}^{(\mathbf{I},s)}(\xi, \eta, \zeta), \quad (u, v, w); \tag{16}$$

– for the second layer:

$$u^{(\mathbf{II},s)}(\xi, \eta, \zeta) = \left(B_2^{(u,s)}(\xi, \eta) - B_4^{(u,s)}(\xi, \eta)\right) \cos \chi^{(\mathbf{I},u)} (\zeta - \zeta_1) -$$

$$- \frac{\tilde{a}_{55}^{\mathbf{II}} \sin \chi^{(\mathbf{II},u)} (\zeta - \zeta_1)}{\tilde{a}_{55}^{\mathbf{I}} \chi^{(\mathbf{II},u)} \cos \chi^{(\mathbf{I},u)} \zeta_1} \times \left(\tilde{a}_{55}^{\mathbf{I}} B_3^{(u,s)}(\xi, \eta) \cos \chi^{(\mathbf{I},u)} \zeta_1 - \right.$$

$$\left. - \chi^{(\mathbf{I},u)} B_1^{(u,s)}(\xi, \eta) + \chi^{(\mathbf{I},u)} B_2^{(u,s)}(\xi, \eta) \sin \chi^{(\mathbf{I},u)} \zeta_1 \right) +$$

$$+ \overline{u}^{(\mathbf{II},s)}(\xi, \eta, \zeta), \quad (u, v, w;\ \ a_{55}^j, a_{44}^j, \frac{\Delta^j}{\Delta_{12}^j}), \tag{17}$$

where

$$B_1^{(u,s)}(\xi, \eta) = \frac{1}{\chi^{(\mathbf{I},u)}} \left(P_u^{(\mathbf{I},s-1)}(\xi, \eta, 0) - \frac{\partial \overline{u}^{(\mathbf{I},s)}(\xi, \eta, 0)}{\partial \zeta} \right),$$

$$B_2^{(u,s)}(\xi, \eta) = u^{+(\mathbf{I},s)}(\xi, \eta, \zeta_1) - \overline{u}^{(\mathbf{I},s)}(\xi, \eta, \zeta_1),$$

$$B_3^{(u,s)}(\xi,\eta) = \frac{1}{\tilde{a}_{55}^{\mathbf{II}}}\left(\frac{\partial \overline{u}^{(\mathbf{II},s)}(\xi,\eta,\zeta_1)}{\partial\zeta} - P_u^{(\mathbf{II},s-1)}(\xi,\eta,\zeta_1)\right) -$$

$$-\frac{1}{\tilde{a}_{55}^{\mathbf{I}}}\left(\frac{\partial \overline{u}^{(\mathbf{I},s)}(\xi,\eta,\zeta_1)}{\partial\zeta} - P_u^{(\mathbf{I},s-1)}(\xi,\eta,\zeta_1)\right),$$

$$B_4^{(u,s)}(\xi,\eta) = \overline{u}^{(\mathbf{II},s)}(\xi,\eta,\zeta_1) - \overline{u}^{(\mathbf{I},s)}(\xi,\eta,\zeta_1)\,,$$

$$u^{+(\mathbf{I},0)} = U^+/R, \quad u^{+(\mathbf{I},s)} = 0, \quad s > 0\,,$$

$$(u, v, w;\ \ a_{55}^j, a_{44}^j, \frac{\Delta^j}{\Delta_{12}^j})\,.$$

The stress tensor components may now be calculated by (11).

In particular, the resulting two-term approximation for the displacements may be presented as

$$U^{(\mathbf{I})}(\xi,\eta,\zeta) = \frac{1}{\cos\chi^{(\mathbf{I},u)}\zeta_1}\left((U^+ - h\overline{u}^{(\mathbf{I},1)}(\xi,\eta,\zeta_1))\cos\chi^{(\mathbf{I},u)}\zeta + \right.$$

$$\left. + \frac{h}{\chi^{(\mathbf{I},u)}}B_1^{(u,1)}(\xi,\eta)\sin\chi^{(\mathbf{I},u)}(\zeta-\zeta_1)\right) +$$

$$+ h\overline{u}^{(\mathbf{I},1)}(\xi,\eta,\zeta), \quad (u, v, w)\,,$$

$$U^{(\mathbf{II})}(\xi,\eta,\zeta) = \left[(U^+ - h\overline{u}^{(\mathbf{II},1)}(\xi,\eta,\zeta_1))\cos\chi^{(\mathbf{I},u)}(\zeta-\zeta_1) + \right.$$

$$+ \frac{a_{55}^{\mathbf{II}}}{a_{55}^{\mathbf{I}}}\left(\frac{U^+\chi^{(\mathbf{I},u)}\sin\chi^{(\mathbf{I},u)}\zeta_1}{\chi^{(\mathbf{II},u)}} + ha_{55}^{\mathbf{I}}B_3^{(u,1)}(\xi,\eta)\cos\chi^{(\mathbf{I},u)}\zeta_1 - \right.$$

$$\left. - h\frac{\partial\overline{u}^{(\mathbf{I},1)}(\xi,\eta,0)}{\partial\zeta} - h\chi^{(\mathbf{I},u)}\overline{u}^{(\mathbf{I},1)}(\xi,\eta,\zeta_1)\sin\chi^{(\mathbf{I},u)}\zeta_1\right)\times$$

$$\left.\times\frac{\sin\chi^{(\mathbf{II},u)}(\zeta-\zeta_1)}{\cos\chi^{(\mathbf{I},u)}\zeta_1}\right] + h\overline{u}^{(\mathbf{II},1)}(\xi,\eta,\zeta),$$

$$(u, v, w;\ \ a_{55}^j, a_{44}^j, \frac{\Delta^j}{\Delta_{12}^j})\,. \qquad (18)$$

It should be noted that the relation $\cos\chi^{(\mathbf{I},u)}\zeta_1 = 0$, (u, v, w), in (15) coincides with the resonances for a single-layered elastic shell in the case of classical boundary conditions, see e.g. [2], i.e. when the upper surface is free and the time-harmonic displacement is prescribed at the lower face. The same conditions are associated with thickness resonances, and moreover, it is known that in the vicinity of these resonances the asymptotic solution degenerates, requiring a special treatment, see [5, 16].

2. Forced vibrations of a three-layered elastic shell with prescribed displacements on the contact surface between the first and second layers. Consider a three-layered elastic orthotropic shell occupying the domain $D = \{\alpha, \beta, \gamma;\ \alpha, \beta \in D_0,\ 0 \le \gamma \le h_1 + h_2 + h_3\}$, where as above D_0 is the upper surface of the first layer, and α, β are lines of curvature of the surface D_0. The governing equations of elastodynamics are given by (1), (2), where now $j = \mathbf{I}, \mathbf{II}, \mathbf{III}$. The boundary and contact conditions are as follows: the upper face $\gamma = 0$ is traction-free (3), and the layers are assumed to be in perfect contact:

$$\tau_{\alpha\gamma}^{\mathbf{I}}(\alpha,\beta,h_1,t) = \tau_{\alpha\gamma}^{\mathbf{II}}(\alpha,\beta,h_1,t), \quad (\tau_{\alpha\gamma},\tau_{\beta\gamma},\tau_{\gamma\gamma}),$$

$$U^{\mathbf{I}}(\alpha,\beta,h_1,t) = U^{\mathbf{II}}(\alpha,\beta,h_1,t), \quad (U,V,W), \tag{19}$$

$$\tau_{\alpha\gamma}^{\mathbf{II}}(\alpha,\beta,h_1+h_2,t) = \tau_{\alpha\gamma}^{\mathbf{III}}(\alpha,\beta,h_1+h_2,t), \quad (\tau_{\alpha\gamma},\tau_{\beta\gamma},\tau_{\gamma\gamma}),$$

$$U^{\mathbf{II}}(\alpha,\beta,h_1+h_2,t) = U^{\mathbf{III}}(\alpha,\beta,h_1+h_2,t), \quad (U,V,W), \tag{20}$$

with time-harmonic displacement field (4) prescribed at the contact surface between the first and second layers.

The solution procedure is pretty similar to that presented in the previous Section 1. Satisfying the conditions (3), (4), (19), (20), the resulting displacements $u^{(\mathbf{I},s)}(\xi,\eta,\zeta)$ and $u^{(\mathbf{II},s)}(\xi,\eta,\zeta)$ for the first and second layers coincide with (16) and (17), respectively, whereas the displacements for the third layer are

$$u^{(\mathbf{III},s)}(\xi,\eta,\zeta) = \frac{1}{\tilde{a}_{55}^{I}\chi^{(\mathbf{II},u)}\cos\chi^{(\mathbf{I},u)}\zeta_1}\Big(M_1^{(u,s)}(\xi,\eta)\tilde{a}_{55}^{\mathbf{II}}\chi^{(\mathbf{III},u)}\times$$
$$\times\cos\chi^{(\mathbf{III},u)}(\zeta-\zeta_2) - T_1^{(u,s)}(\xi,\eta)\tilde{a}_{55}^{\mathbf{III}}\chi^{(\mathbf{II},u)}\times$$
$$\times\sin\chi^{(\mathbf{III},u)}(\zeta-\zeta_2)\Big) + \overline{u}^{(\mathbf{III},s)}(\xi,\eta,\zeta),$$

$$(u,v,w;\ \ a_{55}^{j},a_{44}^{j},\frac{\Delta^{j}}{\Delta_{12}^{j}}), \quad \zeta_2 = \frac{h_1+h_2}{h}, \tag{21}$$

where

$$M_1^{(u,s)}(\xi,\eta) = -\tilde{a}_{55}^{I}\chi^{(\mathbf{II},u)}B_5^{(u,s)}(\xi,\eta) + \tilde{a}_{55}^{\mathbf{I}}\Big(B_2^{(u,s)}(\xi,\eta) -$$
$$- B_4^{(u,s)}(\xi,\eta)\Big)\chi^{(\mathbf{II},u)}\cos\chi^{(\mathbf{II},u)}(\zeta_1-\zeta_2) +$$
$$+ \tilde{a}_{55}^{\mathbf{II}}\sin\chi^{(\mathbf{II},u)}(\zeta_1-\zeta_2)\Big(\ \tilde{a}_{55}^{\mathbf{I}}\chi^{(\mathbf{III},u)} -$$
$$- B_1^{(u,s)}(\xi,\eta)\chi^{(\mathbf{I},u)}/\cos\chi^{(\mathbf{I},u)}\zeta_1 +$$
$$+ B_2^{(u,s)}(\xi,\eta)\chi^{(\mathbf{I},u)}\ \mathrm{tg}\,\chi^{(\mathbf{I},u)}\zeta_1\Big),$$

$$T_1^{(u,s)}(\xi,\eta) = \tilde{a}_{55}^{\mathbf{I}}\tilde{a}_{55}^{\mathbf{II}}B_6^{(u,s)}(\xi,\eta) + \tilde{a}_{55}^{\mathbf{I}}\Big(B_4^{(u,s)}(\xi,\eta) -$$
$$- B_2^{(u,s)}(\xi,\eta)\Big)\chi^{(\mathbf{II},u)}\sin\chi^{(\mathbf{II},u)}(\zeta_1-\zeta_2) +$$
$$+ \tilde{a}_{55}^{\mathbf{II}}\cos\chi^{(\mathbf{II},u)}(\zeta_1-\zeta_2)\Big(\ \tilde{a}_{55}^{\mathbf{I}}\chi^{(\mathbf{III},u)} -$$
$$- B_1^{(u,s)}(\xi,\eta)\chi^{(\mathbf{I},u)}/\cos\chi^{(\mathbf{I},u)}\zeta_1 + B_2^{(u,s)}(\xi,\eta)\chi^{(\mathbf{I},u)}\ \mathrm{tg}\,\chi^{(\mathbf{I},u)}\zeta_1\Big),$$

$$B_5^{(u,s)}(\xi,\eta) = \overline{u}^{(\mathbf{III},s)}(\xi,\eta,\zeta_2) - \overline{u}^{(\mathbf{II},s)}(\xi,\eta,\zeta_2),$$

and

$$B_6^{(u,s)}(\xi,\eta) = \frac{1}{\tilde{a}_{55}^{\mathbf{III}}}\left(\frac{\partial \overline{u}^{(\mathbf{III},s)}(\xi,\eta,\zeta_2)}{\partial\zeta} - P_u^{(\mathbf{III},s-1)}(\xi,\eta,\zeta_2)\right) -$$

$$-\frac{1}{\tilde{a}_{55}^{II}}\left(\frac{\partial \overline{u}^{(\mathbf{II},s)}(\xi,\eta,\zeta_2)}{\partial\zeta} - P_u^{(\mathbf{II},s-1)}(\xi,\eta,\zeta_2)\right),$$

$$(u,v,w;\ \ a_{55}^j, a_{44}^j, \frac{\Delta^j}{\Delta_{12}^j}).$$

The solutions will be finite provided there is no resonance, i.e.

$$\cos\chi^{(\mathbf{I},u)}\zeta_1 \neq 0, \qquad \zeta_1 = h_1/h, \qquad (u,v,w). \tag{22}$$

The two-term resulting displacements for the first and second layers coincide with (18), whereas for the third layer

$$U^{(\mathbf{III})}(\xi,\eta,\zeta) = \left[\left(RM_1^{(u,0)}(\xi,\eta) + hM_1^{(u,1)}(\xi,\eta)\right)a_{55}^{\mathbf{II}}\chi^{(\mathbf{III},u)} \times\right.$$

$$\times\cos\chi^{(\mathbf{III},u)}(\zeta-\zeta_2) - \left(RT_1^{(u,0)}(\xi,\eta) + hT_1^{(u,1)}(\xi,\eta)\right)\times$$

$$\left.\times\, a_{55}^{\mathbf{III}}\chi^{(\mathbf{II},u)}\sin\chi^{(\mathbf{III},u)}(\zeta-\zeta_2)\right] \Big/ \left(a_{55}^{\mathbf{I}}\chi^{(\mathbf{II},u)}\cos\chi^{(\mathbf{I},u)}\zeta_1\right) +$$

$$+ h\overline{u}^{(\mathbf{III},1)}(\xi,\eta,\zeta), \quad (u,v,w;\ \ a_{55}^j, a_{44}^j, \frac{\Delta^j}{\Delta_{12}^j}). \tag{23}$$

The stresses may be calculated using (11). Note that the conditions (22) predictably coincide with (15) for a two-layered shell, and the resonant conditions involve only the parameters of the first layer, whereas the overall stress-strain state is obviously affected by the parameters of all three layers.

3. Forced vibrations of a three-layered elastic shell with prescribed displacements on the interface of the second and third layers. Consider now the situation when for the same three-layered orthotropic elastic shell $D = \{\alpha,\beta,\gamma;\ \alpha,\beta\in D_0,\ 0\le\gamma\le h_1+h_2+h_3\}$, the displacements are imposed on the contact surface between the second and third layers :

$$U^{\mathbf{II}}(\alpha,\beta,h_1+h_2,t) = U^{\mathbf{III}}(\alpha,\beta,h_1+h_2,t) = U^{+}(\alpha,\beta)\exp(i\Omega t),$$

$$(U,V,W). \tag{24}$$

As before, conditions (19), (20) of perfect contact between the layers are supposed, and the upper face $\gamma = 0$ is traction-free, i.e. conditions (3) hold.

Satisfying the boundary conditions (3), (24), (19), (20), we get the solutions for displacements

for the first layer:

$$u^{(\mathbf{I},s)}(\xi,\eta,\zeta) = \frac{1}{\Delta_{u3}}\left(\tilde{a}_{55}^{\mathbf{I}}(\chi^{(\mathbf{II},u)}B_{21}^{(u,s)}(\xi,\eta) +\right.$$

$$+\chi^{(\mathbf{II},u)}B_4^{(u,s)}(\xi,\eta)\cos\chi^{(\mathbf{II},u)}(\zeta_1-\zeta_2) -$$

$$-\tilde{a}_{55}^{\mathbf{II}}B_3^{(u,s)}(\xi,\eta)\sin\chi^{(\mathbf{II},u)}(\zeta_1-\zeta_2))\cos\chi^{(\mathbf{I},u)}\zeta +$$

$$+\chi^{(\mathbf{II},u)}\tilde{a}_{55}^{\mathbf{I}}B_1^{(u,s)}(\xi,\eta)\cos\chi^{(\mathbf{II},u)}(\zeta_1-\zeta_2)\sin\chi^{(\mathbf{I},u)}(\zeta-\zeta_1) +$$

$$+\tilde{a}_{55}^{\mathbf{II}}B_1^{(u,s)}(\xi,\eta)\chi^{(\mathbf{I},u)}\sin\chi^{(\mathbf{II},u)}(\zeta_1-\zeta_2)\cos\chi^{(\mathbf{I},u)}(\zeta-\zeta_1)\Big)+$$

$$+\,\overline{u}^{(\mathbf{I},s)}(\xi,\eta,\zeta),\quad (u,v,w)\,, \tag{25}$$

where

$$B_{21}^{(u,s)}(\xi,\eta)=u^{+(\mathbf{II},s)}(\xi,\eta,\zeta_2)-\overline{u}^{(\mathbf{II},s)}(\xi,\eta,\zeta_2),\quad (u,v,w)\,.$$

for the second layer:

$$u^{(\mathbf{II},s)}(\xi,\eta,\zeta)=\frac{1}{\Delta_{u3}}\Big(\tilde{a}_{55}^{\mathbf{I}}\chi^{(\mathbf{II},u)}B_{21}^{(u,s)}(\xi,\eta)\cos\chi^{(\mathbf{I},u)}\zeta_1\cos\chi^{(\mathbf{II},u)}(\zeta-\zeta_1)-$$

$$-\,\tilde{a}_{55}^{\mathbf{II}}B_{21}^{(u,s)}(\xi,\eta)\chi^{(\mathbf{I},u)}\sin\chi^{(\mathbf{I},u)}\zeta_1\sin\chi^{(\mathbf{II},u)}(\zeta-\zeta_1)-$$

$$-\,\tilde{a}_{55}^{\mathbf{II}}\big(-B_1^{(u,s)}(\xi,\eta)\chi^{(\mathbf{I},u)}+\tilde{a}_{55}^{\mathbf{I}}B_3^{(u,s)}(\xi,\eta)\big)\cos\chi^{(\mathbf{I},u)}\zeta_1+$$

$$+\,B_4^{(u,s)}(\xi,\eta)\chi^{(\mathbf{I},u)}\sin\chi^{(\mathbf{I},u)}\zeta_1\Big)\sin\chi^{(\mathbf{II},u)}(\zeta-\zeta_2)+$$

$$+\,\overline{u}^{(\mathbf{II},s)}(\xi,\eta,\zeta),\quad \Big(u,v,w;\;\; a_{55}^j,a_{44}^j,\frac{\Delta^j}{\Delta_{12}^j}\Big)\,, \tag{26}$$

and for the third layer:

$$u^{(\mathbf{III},s)}(\xi,\eta,\zeta)=\frac{\tilde{a}_{55}^{\mathbf{II}}}{\chi^{(\mathbf{III},u)}\Delta_{u3}}\Big(P_1^{(u,s)}(\xi,\eta)\sin\chi^{(\mathbf{III},u)}(\zeta-\zeta_2)+$$

$$+\,Q_1^{(u,s)}(\xi,\eta)\cos\chi^{(\mathbf{III},u)}(\zeta-\zeta_2)\Big)+\overline{u}^{(\mathbf{III},s)}(\xi,\eta,\zeta)\,,$$

$$\Big(u,v,w;\;\; a_{55}^j,a_{44}^j,\frac{\Delta^j}{\Delta_{12}^j}\Big)\,, \tag{27}$$

where

$$Q_1^{(u,s)}(\xi,\eta)=\tilde{a}_{55}^{\mathbf{II}}\chi^{(\mathbf{III},u)}\Big(B_{21}^{(u,s)}(\xi,\eta)-B_5^{(u,s)}(\xi,\eta)\Big)\times$$

$$\times\Big(\tilde{a}_{55}^{\mathbf{I}}\chi^{(\mathbf{I},u)}\cos\chi^{(\mathbf{I},u)}\zeta_1\cos\chi^{(\mathbf{II},u)}(\zeta_1-\zeta_2)+$$

$$+\,\tilde{a}_{55}^{\mathbf{II}}\chi^{(\mathbf{I},u)}\sin\chi^{(\mathbf{I},u)}\zeta_1\sin\chi^{(\mathbf{II},u)}(\zeta_1-\zeta_2)\Big),$$

$$P_1^{(u,s)}(\xi,\eta)=\tilde{a}_{55}^{\mathbf{I}}\tilde{a}_{55}^{\mathbf{III}}\chi^{(\mathbf{II},u)}\cos\chi^{(\mathbf{I},u)}\zeta_1\Big(-\tilde{a}_{55}^{\mathbf{II}}B_3^{(u,s)}(\xi,\eta)-$$

$$-\,\tilde{a}_{55}^{\mathbf{II}}B_6^{(u,s)}(\xi,\eta)\cos\chi^{(\mathbf{II},u)}(\zeta_1-\zeta_2)+$$

$$+\,B_{21}^{(u,s)}(\xi,\eta)\chi^{(\mathbf{II},u)}\sin\chi^{(\mathbf{II},u)}(\zeta_1-\zeta_2)\Big)+$$

$$+\,\tilde{a}_{55}^{\mathbf{II}}\tilde{a}_{55}^{\mathbf{III}}\chi^{(\mathbf{I},u)}\Big(B_1^{(u,s)}(\xi,\eta)\chi^{(\mathbf{II},u)}-$$

$$-\sin\chi^{(\mathbf{I},u)}\zeta_1\big(B_4^{(u,s)}(\xi,\eta)\chi^{(\mathbf{II},u)}+$$

$$+\,B_{21}^{(u,s)}(\xi,\eta)\chi^{(\mathbf{II},u)}\cos\chi^{(\mathbf{II},u)}(\zeta_1-\zeta_2)+$$

$$+\,\tilde{a}_{55}^{\mathbf{II}}B_6^{(u,s)}(\xi,\eta)\sin\chi^{(\mathbf{II},u)}(\zeta_1-\zeta_2)\big)\Big).$$

The non-resonant condition allowing the finite solution is given by

$$\Delta_{u3} = \tilde{a}_{55}^{\mathbf{I}} \chi^{(\mathbf{II},u)} \cos \chi^{(\mathbf{I},u)} \zeta_1 \cos \chi^{(\mathbf{II},u)} (\zeta_1 - \zeta_2) +$$

$$+ \tilde{a}_{55}^{\mathbf{II}} \chi^{(\mathbf{I},u)} \sin \chi^{(\mathbf{I},u)} \zeta_1 \sin \chi^{(\mathbf{II},u)} (\zeta_1 - \zeta_2) \neq 0 ,$$

$$(u, v, w; \ \ a_{55}^j, a_{44}^j, \frac{\Delta^j}{\Delta_{12}^j}) . \tag{28}$$

Once again, the stresses may now be calculated from (11).

It is worth noting that formulae (28) in fact coincides with the corresponding non-resonant conditions for a two-layered orthotropic shell within the conventional formulation, see [1], i.e. when the upper face is free, and the excitation in the form of associated displacements is applied at the lower face (provided the layers are in perfect contact).

Once again, we observe that relations (28) imply that the parameters of the first two layers are only involved in the resonant conditions, whereas the overall stress-strain state includes the parameters of all three layers. Thus, as expected, the resonant conditions involve only the parameters of the layers above the contact surface at which the displacements were prescribed.

Once the stresses and strains are determined, the potential energy of the deformation can be determined by a well-known formula

$$E = \frac{1}{2} \int_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\varepsilon_{xy} + \sigma_{xz}\varepsilon_{xz} + \sigma_{yz}\varepsilon_{yz}) \, dv , \tag{29}$$

whereas the relation

$$\log E = 11.8 + 1.5M \tag{30}$$

gives estimate for the magnitude M of the anticipated earthquake, see [19] for more details.

Conclusion. In this paper, the asymptotic solutions of 3D dynamic problems for multi-layered orthotropic elastic shells with a traction-free upper face and prescribed displacements on one of the contact inter-layer surfaces have been obtained in assumption of perfect contact between the layers. The solutions for displacements and stresses have been derived in the form of asymptotic series along the small geometrical parameter, which is a natural feature of thin shells.

The results may be applied to estimate the possibility of delamination, as well as to calculate the deformation energy required for seismological applications [19].

The developed methodology may be extended to transient vibrations. Another potential development is related to analysis of shells of finite size, in which accounting for static and, in general, dynamic boundary layers would be required, for more details see [4]. The proposed technique may also be implemented for the case of non-perfect contact between the layers, as well as a shell laminate containing more layers. Finally, we not a less trivial generalisation for layered shells with layers of variable thickness.

Acknowledgement. LGG acknowledges support by the Erasmus+ICM program within the framework of collaboration between ASPU after Kh. Abovyan, Armenia, and Keele University, UK.

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ТРИВИМІРНИЙ ДИНАМІЧНИЙ АНАЛІЗ ШАРУВАТИХ ПРУЖНИХ ОБОЛОНОК

Розглянуто тривимірні динамічні задачі для шаруватих ортотропних пружних оболонок із вільною верхньою поверхнею і заданими переміщеннями на одній із меж поділу шарів за умови ідеального контакту. Побудовано довгохвильовий асимптотичний розв'язок і визначено резонанси товщини. Отримані результати можуть знайти подальше застосування при оцінці певних параметрів землетрусів.

Ключові слова: *пружна 3-D динаміка, шарувата оболонка, асимптотичний метод.*

[1] Institute of Mechanics of NAS RA, Armenia,
[2] Kh. Abovyan Armenian State Pedagogical University, Armenia,
[3] School of Computing and Mathematics, Keele University, Keele, UK

Received
15.12.20