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TROPICAL SEMIRINGS OF LIPSCHITZ FUNCTIONS

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A function f from a metric space (M_1, ρ_1) to a metric space (M_2, ρ_2) is *Lipschitz* if there is a constant L such that

$$\rho(f(x), f(y)) \leq L\rho(x, y), \quad x, y \in M_2.$$

Our basic object of study is the tropical semiring $(R \cup \infty, \bigoplus, \bigodot)$ [1]. As a set this is just the real numbers R, together with an extra element ∞ that represents infinity. However, we redefine the basic arithmetic operations of addition and multiplication of real numbers as follows:

$$x \bigoplus y := min(x, y) \qquad \text{and} \qquad x \bigodot y := x + y.$$

A *tropical polynomial* of several variables $t_1, \ldots, t_n$ in $\mathbb{R} \cup \{+\infty\}$ is a function of the form

$$p(t_1, \ldots, t_n) = a \odot t_1^{i_1} \cdots t_n^{i_n} \oplus b \odot t_1^{j_1} \cdots t_n^{j_n} \oplus \cdots$$
$$= \min(a + i_1 t_1 + \cdots + i_n t^n, b + j_1 t_1 + \cdots + j_n t^n, \cdots),$$

where the coefficients $a, b, \ldots$ are real numbers and the exponents $i_1, j_1, \ldots$ are *integers*. We can see that any tropical polynomial can be represented as minimum of some affine functions. Hence, every tropical polynomial is a Lipschitz function on $\mathbb{R}^n$ and a finite composition of tropical polynomials is a Lipschitz function. Note, that a composition of tropical polynomials is not a tropical polynomial in the general case.

Theorem 1. *Let $g_{\varphi_1}, \ldots, g_{\varphi_n}$ be Lipschitz functions on X, generated by functionals $\varphi_1, \ldots, \varphi_n \in X(\mathbb{Z}_0)^*$ where $g_\varphi(x) := \sum_{n \in \mathbb{Z}_0} c_n g_n(x)$ and $q(t_1, \ldots, t_n)$ be a finite composition of tropical polynomials of variables $t_1, \ldots, t_n$. Then $Q(x) = q(g_1(x), \ldots, g_n(x)), \quad x \in X$, is a Lipschitz function on X.*

1. *Speyer D. E., Sturmfels B.* Tropical mathematics // Math. Mag. – 2009. – **82**. – P. 163–173. doi:10.1080/0025570X.2009.11953615

http://iapmm.lviv.ua/mpmm2023/materials/ma08_07.pdf