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INFINITESIMAL OPERATOR FOR THE CONTINUOUS TIME MARKOV MODULATED POISSON PROCESS

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Markov Modulated Poisson Processes (MMPP) [1] are commonly used to capture the burstiness and temporal correlation in network traffic patterns (in network traffic modeling), to describe the volatility of financial asset prices (in financial modeling), to model the spread of infectious diseases and to analyze the effectiveness of various interventions (in epidemiology). Alongside, it can be used for modeling an intrusion detection system (IDS) by representing the behavior of a network traffic as a sequence of states, where each state is associated with a Poisson arrival rate.

Let's consider MMPP as a two-component process $(x(t), N(t))$. Here, the process $x(t) = x_t, t \geq 0$ is uniformly ergodic Markov process in the standard phase space $(X, \boldsymbol{X})$ with infinitesimal operator (generator) [2]:

$$Q\varphi(x) = q(x) \int_X P(x, dy) \left[\varphi(y) - \varphi(x)\right], \quad \varphi \in \mathbf{B}(X), \tag{1}$$

where $\mathbf{B}$ denotes Banach space of real bounded functions with supremum norm $||\varphi|| = \max_{x \in X} |\varphi(x)|$. Let process $x(t)$ have stationary distribution $\pi(C)$, $C \in \boldsymbol{X}$.

The Poisson process $N(t) = N_t, t \geq 0$ is defined as the counting process of the number of events that have occurred up to time t. The arrival rate of events at time t is given by $\lambda(x(t))$, i.e. rate parameter is modulated by the state process.

In other words, the continuous-time Markov-modulated Poisson process is a stochastic process that counts the number of events that occur in a Poisson process, where the arrival rate of events is modulated by a Markovian state process.

The probability of a random variable $N(t)$ being equal to k is given by:

$$P[N(t) = k] = \frac{\Lambda^k(t)}{k!} e^{-\Lambda(t)}, \tag{2}$$

where $\Lambda(t) = \int_0^t \lambda(x(t))dt$ is the cumulative arrival rate up to time t, where $\lambda(x(t))$ is the arrival rate of events at time t when the Markov process is in state $x(t)$. For such a process the following holds true:

$$\begin{aligned} &P[N(t+h) - N(t) = 0] = 1 - \Lambda(h) + o(h), \\ &P[N(t+h) - N(t) = 1] = \Lambda(h) + o(h), \\ &P[N(t+h) - N(t) > 1] = o(h), h \to 0. \end{aligned} \tag{3}$$

Lemma 1. *Infinitesimal operator for the continuous time Markov Modulated Poisson Process (x_t, N_t) has the form:*

$$L\varphi(x, N) = Q\varphi(x, N) + \lambda(x)(R\varphi(x, N) - I\varphi(x, N)), \tag{4}$$

where $R\varphi(x, N) = \varphi(x, N+1)$ and $I\varphi(x, N) = \varphi(x, N)$ - an identity operator.

Proof. Similar to [3] the definition of the infinitesimal operator is used. Then the conditional mathematical expectation is calculated and with the usage of the Poisson process properties and Markov process generator definition (1) the resulting formula (4) is obtained. ■

The infinitesimal operator is useful for analyzing the statistical properties of the process, such as its moments, correlation functions, and steady-state behavior. It can also be used to derive the stochastic differential equations that describe the dynamics of systems that are perturbed by MMPP noise.

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http://iapmm.lviv.ua/mpmm2023/materials/mm11_04.pdf