TORSIONAL CRACK HEALING IN A NONHOMOGENEOUS CYLINDER

Victor Sylovanyuk, Maksym Filipov

Karpenko Physico-Mechanical Institute of the NAS of Ukraine, Lviv
vsylovanyuk@gmail.com, maksym.filipov@gmail.com

Three-dimensional problems of crack theory for nonhomogeneous materials are associated with significant mathematical difficulties. For a number of nonhomogeneous materials, the inhomogeneity can be adequately expressed by some coordinate function for one of the elastic moduli, for example, the shear modulus, while the Poisson's ratio can remain constant. This assumption makes it possible to simplify the mathematical model.

Suppose that an isotropic continuously nonhomogeneous cylinder of radius R contains a plane circular crack of radius a. Let us place the coordinate system (r,θ,z) so that the crack occupies the area $z=0$, $0\leqslant r\leqslant a$.

We will assume that the shear modulus μ is a function of the radial coordinate of the following form:

$$\mu(r)=\mu_0 r^m, \quad m\geq 0, \quad \mu_0=\text{const} \tag{1}$$

Suppose the cylinder is twisted so that the deformations are uniform along lines parallel to the z-axis and only the displacements u_θ are non-zero. Then the stress field is defined by the relations

$$\tau_{\theta z}=\mu\frac{\partial u_\theta}{\partial z}, \quad \tau_{\theta r}=\mu r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) \tag{2}$$

The equation of equilibrium of the body in this case is as follows:

$$\frac{\partial \tau_{r\theta}}{\partial r}+\frac{\partial \tau_{z\theta}}{\partial z}+\frac{2\tau_{r\theta}}{r}=0, \tag{3}$$

or in displacements

$$\frac{\partial^2 u_\theta}{\partial z^2}+\frac{\partial^2 u_\theta}{\partial r^2}+(1+m)\frac{\partial}{\partial z}\left(\frac{u_\theta}{r}\right)=0\,. \tag{4}$$

To strengthen the body, the crack was filled with an injected material that, after polymerization or crystallization, forms strong adhesive bonds with the cylinder material. Let us determine the effectiveness of this kind of strengthening technology. The conditions of contact of the formed elastic layer with the base material under torsion will be presented in accordance with the Winkler-type shear hypothesis.

If the crack dimensions are small compared to the cylinder diameter ($R \gg a$), then, given the symmetry of the problem with respect to the plane $z=0$, it is sufficient to consider the equilibrium of the half-space $z\geq 0$ with boundary conditions

$$\tau_{z\theta}(r,0)=\tau(r)+u_\theta(r)\mu_*/h(r), 0\leqslant r\leqslant a;\ u_\theta(r,0)=0,\ r>a.. \quad (5)$$

Here, μ_* is the shear modulus of the crack filler material; $2h(r)$ is the thickness of the layer; $\tau(r)$ is a function describing the distribution of shear stresses on the crack surface. The solution of Equation (4) for the half-space $z\geq 0$ using the Hankel integral transform can be obtained as

$$u_\theta(r,z)=\frac{1}{r^{m/2}}\int_0^\infty A(\xi)\exp(-\xi z)J_{1+m/2}(r\xi)d\xi, \quad (6)$$

where J_n is a Bessel function of the first kind; $A(\xi)$ is an unknown function that is determined by the boundary conditions.

On the basis of (2), (5), (6), we find a system of dual integral equations:

$$\int_0^\infty \xi A(\xi)J_{1+m/2}(r\xi)d\xi=-\frac{1}{\mu_0 r^{m/2}}\left(\tau(r)+\frac{u_\theta(r)\mu_*}{r^{m/2}h(r)}\right),\ 0\leqslant r\leqslant a,$$
$$\int_0^\infty A(\xi)J_{1+m/2}(r\xi)d\xi=0,\ r>a. \quad (7)$$

The solution of the system of dual integral equations can be given as

$$A(\xi)=-\frac{1}{\mu_0}\left(\frac{2\xi}{\pi}\right)^{1/2}\int_0^a t^{-\frac{1+m}{2}}J_{\frac{3+m}{2}}(\xi t)dt\int_0^t \frac{\rho^2}{\sqrt{t^2-\rho^2}}\left(\tau(\rho)+\frac{\mu_* u_\theta(\rho)}{h(\rho)}\right)d\rho. \quad (8)$$

The unknown displacements of the crack surfaces u_θ are determined by taking into account Eqs. (6) and (8). As a result, we obtain the Fredholm integral equations of the second kind with respect to u_θ:

$$u_\theta(r)+\frac{\mu_*}{r^{m/2}\mu_0}\left(\frac{2}{\pi}\right)^{1/2}\int_0^a \frac{\rho^2 u_\theta(\rho)d\rho}{h(\rho)}\int_\rho^a \frac{t^{-1+m/2}dt}{\sqrt{t^2-\rho^2}}\int_0^\infty \sqrt{\xi}J_{1+m/2}(\xi r)J_{(3+m)/2}(\xi t)d\xi$$
$$=-\frac{1}{r^{m/2}\mu_0}\left(\frac{2}{\pi}\right)^{1/2}\int_0^a t^{-1+m/2}dt\int_0^t \frac{\rho^2\tau(\rho)d\rho}{\sqrt{t^2-\rho^2}}\int_0^\infty \sqrt{\xi}J_{1+m/2}(\xi r)J_{(3+m)/2}(\xi t)d\xi.$$

If one of the methods for solving Fredholm's integral equations of the second kind is to establish the function u_θ, then the stress intensity factor k_3 can be calculated by the formula

$$k_3=-\frac{2}{\pi\sqrt{a}}\int_0^a \frac{r^2}{\sqrt{a^2-r^2}}\left(\tau(r)+\frac{\mu_* u_\theta(r)}{h(r)}\right)dr.$$

As can be seen, for an unfilled crack, a change in the shear modulus in the radial direction does not affect the intensity of the stress field near the top of the defect. For a healed crack, the value of k_3 depends on the variable shear modulus (1).